Chapter 3. THE EXISTENCE OF OPTIMAL STOPPING RULES.

Consider the general stopping rule problem of Chapter 1 with observations $X_1, X_2, \ldots$ and rewards $Y_0, Y_1, \ldots, Y_\infty$ where $Y_n = y_n(X_1, \ldots, X_n)$. The following two assumptions are basic to the theory of this chapter.

A1. $\mathrm{E}\{\sup_n Y_n\} < \infty$.
A2. $\limsup_{n\to\infty} Y_n \le Y_\infty$ a.s.

Assumption A1 allows us to interchange expectation and summation in what follows. It implies that even a prophet who can foresee the future and stop at the time that Y_n assumes its maximum value, or comes close to the supremum if the maximum does not exist, can only obtain a finite expected return. Thus, certainly $\sup_N \mathrm{E}Y_N < \infty$, where the supremum is taken over all stopping rules, N.

In §3.1, we show that under these two assumptions an optimal stopping rule exists. The treatment follows the method of Chow and Robbins (1963) using the notion of a *regular* stopping rule. In §3.2, we discuss the principle of optimality and the optimality equation. We show under assumptions A1 and A2 that the rule given by the principle of optimality is optimal. In §3.3, we derive Wald's equation, and in §3.4, we examine prophet inequalities.

Here are two examples that show if either one of the assumptions is not satisfied, an optimal stopping rule may not exist.

EXAMPLE 1. Let $X_1, X_2, \ldots$ be independent Bernoulli trials with probability 1/2 of success, and let $Y_0 = 0$,

$$Y_n = (2^n - 1)\prod_1^n X_i, \tag{1}$$

and $Y_\infty = 0$. As long as only successes have occurred, you may stop at stage n and receive $2^n - 1$; after the first failure has occurred, you receive 0. Since $Y_n \to 0$ a.s., A2 is satisfied. On the other hand, $\sup_n Y_n = 2^k - 1$ with probability $1/2^{k+1}$ for $k = 0, 1, 2, \ldots$ so that $\mathrm{E}\{\sup_n Y_n\} = \sum_0^\infty (1 - 1/2^k)/2 = \infty$ and A1 is not satisfied. If you reach stage n without any failures, your return for stopping is $2^n - 1$, while if you continue one stage you can get an expected value of at least $(2^{n+1} - 1)/2 = 2^n - 1/2$, which is better. Thus, it can never be optimal to stop before a failure has occurred. Yet continuing forever gives you a zero payoff so there is no optimal stopping rule. In fact, $\sup_N \mathrm{E}Y_N = 1$, but the supremum is not attained. ■

EXAMPLE 2. Let $Y_0 = 0, Y_n = 1 - 1/n$ for $n = 1, 2, \ldots$ and $Y_\infty = 0$. (The X_n are immaterial.) Here A1 is satisfied and A2 is not. Yet, like the previous example, the longer you wait the better off you are, but if you wait forever you win nothing. There is no optimal rule. ■

We remark that for minimization problems, where Y_n represents a cost rather than a reward, conditions A1 and A2 should be replaced by

A1. $\mathrm{E}\{\inf_n Y_n\} > -\infty$.
A2. $\liminf_{n\to\infty} Y_n \geq Y_\infty$ a.s.

§**3.1. Regular Stopping Rules.** We precede the main theorem on the existence of optimal stopping rules by two lemmas involving the notion of a regular stopping rule, a concept due to Snell (1952).

Definition. *A stopping rule N is said to be* **regular**, *if for every n,*

$$\mathrm{E}\{Y_N|\mathcal{F}_n\} > Y_n \quad \text{a.s. on} \quad \{N > n\}. \tag{2}$$

In other words, N is regular if $\mathrm{E}\{Y_N|X_1 = x_1, \ldots, X_n = x_n\} > y_n(x_1, \ldots, x_n)$ for almost all $(x_1, \ldots, x_n) \in \{N > n\}$. In still other words, N is regular if it has the property that if N tells you to continue at a certain stage, then N gives you an improved conditional expected return compared to stopping at that stage. The first lemma shows how to replace a given stopping rule by a regular one having no worse expected payoff.

Lemma 1. *Assume A1. Given any stopping rule N, there is a regular stopping rule N' such that* $\mathrm{E}Y_{N'} \geq \mathrm{E}Y_N$.

Proof. Define $N' = \min\{n \geq 0 : \mathrm{E}(Y_N|\mathcal{F}_n) \leq Y_n\}$. That is, N' tells you to use N until N tells you to continue and stopping is at least as good, in which case you stop. It is clear that N' is a stopping rule and that $N' \leq N$. On $\{N' = n\}$, we have $\mathrm{E}(Y_N|\mathcal{F}_n) \leq Y_n$ a.s. for all n, while on $\{N' = \infty\}$, we have $Y_N = Y_{N'} = Y_\infty$ a.s. Hence,

$$\begin{aligned} \mathrm{E}Y_{N'} &= \sum_{n=0}^{=\infty} \mathrm{E}(I\{N' = n\}Y_n) \geq \sum_{n=0}^{\infty} \mathrm{E}(I\{N' = n\}\mathrm{E}(Y_N|\mathcal{F}_n)) + \mathrm{E}(I\{N' = \infty\}Y_\infty) \\ &= \sum_{n=0}^{=\infty} \mathrm{E}(I\{N' = n\}Y_N) = \mathrm{E}Y_N, \end{aligned} \tag{3}$$

where the summation is over n from 0 to ∞ inclusive. The interchange of expectation and summation is valid by A1. To see that N' is regular, we use the same argument as above conditional on $\mathcal{F}_n$ to find that on $\{N' > n\}$, we have $\mathrm{E}(Y_{N'}|\mathcal{F}_n) \geq \mathrm{E}(Y_N|\mathcal{F}_n)$ a.s. Since $\mathrm{E}(Y_N|\mathcal{F}_n) > Y_n$ a.s. on $\{N' > n\}$, we have $\mathrm{E}(Y_{N'}|\mathcal{F}_n) > Y_n$ on $\{N' > n\}$. ■

The next lemma shows that we may improve on two regular stopping rules by stopping when the one with the longer life tells us to stop.

Lemma 2. *Assume A1. If N and N' are regular stopping rules, then so is $N'' = \max\{N, N'\}$ and then*

$$\mathrm{E}Y_{N''} \geq \max\{\mathrm{E}Y_N, \mathrm{E}Y_{N'}\}. \tag{4}$$

Proof. N'' is equal to N except on sets of the form $\{N = n\} \cap \{N' > n\}$ in which case $\mathrm{E}(Y_{N''}|\mathcal{F}_n) = \mathrm{E}(Y_{N'}|\mathcal{F}_n) > Y_n$ a.s. Hence,

$$\begin{aligned}\mathrm{E}Y_{N''} &= \sum \mathrm{E}(I\{N = n\}Y_{N''}) = \sum \mathrm{E}(I\{N = n\}\mathrm{E}(Y_{N''}|\mathcal{F}_n)) \\ &\geq \sum \mathrm{E}(I\{N = n\}Y_n) = \mathrm{E}Y_N.\end{aligned} \tag{5}$$

By symmetry, $\mathrm{E}Y_{N''} \geq \mathrm{E}Y_{N'}$.

To see that N'' is regular, note that $\{N'' > n\} = \{N > n\} \cup \{N' > n\}$. On $\{N > n\}$, an argument similar to the above but conditional on $\mathcal{F}_n$ shows that $\mathrm{E}(Y_{N''}|\mathcal{F}_n) \geq \mathrm{E}(Y_N|\mathcal{F}_n)$ a.s., which is greater than Y_n from the regularity of N. By symmetry, $\mathrm{E}(Y_{N''}|\mathcal{F}_n) > Y_n$ a.s. on $\{N' > n\}$. Hence, on $\{N'' > n\}$, $\mathrm{E}(Y_{N''}|\mathcal{F}_n) > Y_n$ a.s. showing the regularity of N''. ∎

With these two lemmas in hand, the main theorem of this chapter is an application of the *Fatou-Lebesgue Lemma*, which states: If $Z, X_1, X_2, \ldots$ is a sequence of real-valued random variables such that $X_n \leq Z$ for all n and $\mathrm{E}Z < \infty$, then $\limsup_n \mathrm{E}X_n \leq \mathrm{E}\limsup_n X_n$.

Theorem 1. *Under A1 and A2, there exists a stopping rule N^* such that $\mathrm{E}Y_{N^*} = V^*$, where $V^* = \sup_N \mathrm{E}Y_N$.*

Proof. If $V^* = -\infty$, the result is trivial. So we may assume that $-\infty < V^* < \infty$. Let $N_1, N_2, \ldots$ be a sequence of stopping rules such that $\mathrm{E}Y_{N_j} \to V^*$. Let $N'_1, N'_2, \ldots$ be the regularized versions as in Lemma 1, so that $\mathrm{E}Y_{N'_j} \to V^*$. Let $N''_j = \max\{N'_1, \ldots, N'_j\}$ so that by Lemma 2, $\mathrm{E}Y_{N''_j} \geq \mathrm{E}Y_{N'_j}$ and consequently $\mathrm{E}Y_{N''_j} \to V^*$. Note that N''_j is a monotone nondecreasing sequence of stopping rules converging to the stopping rule $N^* = \sup\{N'_1, N'_2, \ldots\}$. Moreover, since N''_j is a nondecreasing sequence of integers, either $N''_j \to \infty$ or N''_j is a fixed integer from some j on. Thus, $\limsup_{j\to\infty} Y_{N''_j} \leq Y_{N^*}$ a.s from A2. From the Fatou-Lebesgue Theorem, since the $Y_{N''_j}$ are bounded above by $\sup_n Y_n$ which is integrable by A1,

$$V^* = \limsup \mathrm{E}Y_{N''_j} \leq \mathrm{E}\limsup Y_{N''_j} \leq \mathrm{E}Y_{N^*}. \tag{6}$$

Since $\mathrm{E}Y_{N^*} \leq V^*$ by definition of V^*, we have $\mathrm{E}Y_{N^*} = V^*$. ∎

§**3.2. The Principle of Optimality and the Optimality Equation.** At the initial stage, we can obtain y_0 without sampling, or we can obtain V^* by using an optimal rule. Therefore, it is optimal to stop without sampling if, and only if, $y_0 = V^*$. We expect to be able to apply this principle at later stages too. If we have observed $X_1 = x_1, \ldots, X_n = x_n$, we may obtain $y_n(x_1, \ldots, x_n)$ without sampling further, or we can, by using a rule optimal for the remaining stages, obtain

$$V_n^*(x_1, \ldots, x_n) = \sup_{N \geq n} \mathrm{E}\{Y_N | X_1 = x_1, \ldots, X_n = x_n\}. \tag{7}$$

Note $V_0^* = V^*$. Here $\sup_{N \geq n}$ means supremum over the set of all stopping rules N such that $\mathrm{P}(N \geq n) = 1$. We expect that it will be optimal to stop at stage n having observed $X_1 = x_1, \ldots, X_n = x_n$ if and only if $y_n(x_1, \ldots, x_n) = V_n^*(x_1, \ldots, x_n)$. This is known as the **principle of optimality**. It is also of central importance in the more general dynamic programming problems.

This principle is valid here under A1 and A2 but it requires a modification. The trouble is that in general there are more than a countable number of stopping rules, $N \geq n$, and the supremum of an uncountable collection of random variables (here the supremum of $\mathrm{E}(Y_N | \mathcal{F}_n)$ over the set of those stopping rules $N \geq n$) may not be a random variable (i.e. measurable) and even if it is, it may not be what we want it to be. Thus, (7) is not well defined. We can get around this difficulty by using instead the essential supremum.

Definition. *Let X_t, for $t \in T$, be a collection of random variables. We say that a random variable Z is* **an essential supremum of** $(X_t)_{t \in T}$ *and write $Z = \operatorname{ess\,sup}_{t \in T} X_t$, if*
i. $\mathrm{P}(Z \geq X_t) = 1$ for all $t \in T$, and
ii. if Z' is any other random variable such that $\mathrm{P}(Z' \geq X_t) = 1$ for all $t \in T$, then $\mathrm{P}(Z' \geq Z) = 1$.

As an example of a collection of random variables X_t for which $\operatorname{ess\,sup}_{t \in T} X_t \neq \sup_{t \in T} X_t$, let $T = [0, 1]$ and let $X_t = I(t = U)$ where U is a random variable with a uniform distribution on $[0, 1]$. Then $\sup_{t \in T} X_t = 1$, yet $\operatorname{ess\,sup}_{t \in T} X_t = 0$.

Lemma 3. *An essential supremum, $Z = \operatorname{ess\,sup}_{t \in T} X_t$, always exists, and there exists a countable subset $C \subset T$ such that $Z = \sup_{t \in C} X_t$ is an essential supremum.*

Proof. By taking arctan of the X_t, $(t \in T)$, if necessary, we may assume without loss of generality that the X_t are uniformly bounded. Let $\mathcal{C}$ be the class of all countable subsets of T, and let

$$\alpha = \sup_{S \in \mathcal{C}} \mathrm{E} \sup_{t \in S} X_t. \tag{8}$$

For every $n = 1, 2, \ldots$ find sets $S_n \in \mathcal{C}$ such that $\mathrm{E} \sup_{t \in S_n} X_t \geq \alpha - 1/n$. Let $C = \bigcup_{n=1}^{\infty} S_n$ and $Z = \sup_{t \in C} X_t$ so that C is countable and $\mathrm{E}Z = \alpha$. We now show that $Z = \operatorname{ess\,sup}_{t \in T} X_t$.

i. Suppose for some $t \in T$, $\mathrm{P}(Z < X_t) > 0$. Then $\mathrm{E} \max(Z, X_t) > \mathrm{E}Z \geq \alpha$, contradicting (8). Hence $\mathrm{P}(Z \geq X_t) = 1)$ for all $t \in T$.

ii. Suppose for some Z' that $\mathrm{P}(Z' \geq X_t) = 1$ for all $t \in T$. Then since C is countable, $\mathrm{P}(Z' \geq \sup_{t \in C} X_t) = 1$. Hence $\mathrm{P}(Z' \geq Z) = 1$. ∎

Before returning to the principle of optimality, we first indicate that the analogues of Lemmas 1 and 2 are valid conditionally on having reached stage n. For this purpose, we extend the notion of a regular stopping rule. When we write $X \geq Y$ or $X > Y$ etc., where X and Y are random variables, we take it as implied that the inequalities hold almost surely. Similarly, when we write $X > Y$ on a set A, we mean $\mathrm{P}(\{X > Y\} \cap A) = \mathrm{P}(A)$.

Definition. *A stopping rule* $N \geq n$ *is* **regular from** n **on**, *if for every* $k \geq n$, $\mathrm{E}\{Y_N | \mathcal{F}_k\} > Y_k$ *on* $\{N > k\}$.

Lemma 1′. *Under A1, for any stopping rule* $N \geq n$ *there exists a stopping rule* $N' \geq n$, *regular from* n *on, such that* $\mathrm{E}\{Y_{N'} | \mathcal{F}_n\} \geq \mathrm{E}\{Y_N | \mathcal{F}_n\}$.

Lemma 2′. *Under A1, if* $N \geq n$ *and* $N' \geq n$ *are both regular from* n *on, then so is* $N'' = \max(N, N')$ *and* $\mathrm{E}(Y_{N''} | \mathcal{F}_n) \geq \max(\mathrm{E}\{Y_N | \mathcal{F}_n\}, \mathrm{E}\{Y_{N'} | \mathcal{F}_n\})$.

The proofs of these lemmas are straightforward adaptations of the proofs of Lemmas 1 and 2, and so are omitted.

The next theorem is the **optimality equation** of dynamic programming. Let

$$V_n^* = \operatorname{ess\,sup}_{N \geq n} \mathrm{E}\{Y_N | \mathcal{F}_n\}. \tag{9}$$

Theorem 2. *Under A1,* $V_n^* = \max(Y_n, \mathrm{E}\{V_{n+1}^* | \mathcal{F}_n\})$.

Proof. Let $N \geq n$ be an arbitrary stopping rule. On $\{N > n\}$, $\mathrm{E}\{Y_N | \mathcal{F}_{n+1}\} \leq V_{n+1}^*$, so that on $\{N > n\}$, $\mathrm{E}\{Y_N | \mathcal{F}_n\} = \mathrm{E}\{\mathrm{E}(Y_N | \mathcal{F}_{n+1}) | \mathcal{F}_n\} \leq \mathrm{E}\{V_{n+1}^* | \mathcal{F}_n\}$. Hence $\mathrm{E}\{Y_N | \mathcal{F}_n\} = I\{N = n\} Y_n + I\{N > n\} \mathrm{E}\{Y_N | \mathcal{F}_n\} \leq \max(Y_n, \mathrm{E}\{V_{n+1}^* | \mathcal{F}_n\})$ for all $N \geq n$. Therefore, $V_n^* = \operatorname{ess\,sup}_{N \geq n} \mathrm{E}\{Y_N | \mathcal{F}_n\} \leq \max(Y_n, \mathrm{E}\{V_{n+1}^* | \mathcal{F}_n\})$.

To show the reverse inequality, first note that $Y_n \leq V_n^*$ trivially. Now by Lemma 3, there exists a sequence $N_1, N_2, \ldots$ of stopping rules with each $N_k \geq n+1$ such that $V_{n+1}^* = \sup_k \mathrm{E}\{Y_{N_k} | \mathcal{F}_{n+1}\}$. By Lemma 1′, there exists for each k a stopping rule $N_k' \geq n+1$ regular from $n+1$ on such that $\mathrm{E}\{Y_{N_k'} | \mathcal{F}_{n+1}\} \geq \mathrm{E}\{Y_{N_k} | \mathcal{F}_{n+1}\}$. Let $N_k'' = \max(N_1', \ldots, N_k')$. Then,

$$\begin{aligned} V_n^* &\geq \mathrm{E}\{Y_{N_k''} | \mathcal{F}_n\} = \mathrm{E}(\mathrm{E}\{Y_{N_k''} | \mathcal{F}_{n+1}\} | \mathcal{F}_n) \\ &\geq \mathrm{E}(\max_{1 \leq j \leq k} \mathrm{E}\{Y_{N_j'} | \mathcal{F}_{n+1}\} | \mathcal{F}_n) \quad \text{by Lemma } 2', \\ &\geq \mathrm{E}(\max_{1 \leq j \leq k} \mathrm{E}\{Y_{N_j} | \mathcal{F}_{n+1}\} | \mathcal{F}_n) \\ &\to \mathrm{E}(V_{n+1}^* | \mathcal{F}_n) \end{aligned} \tag{10}$$

by monotone convergence. ∎

It is interesting to note that the optimality equation holds under A1 alone, even if an optimal rule does not exist.

The stopping rule given by the principle of optimality is the rule

$$N^* = \min\{n \geq 0 : Y_n = V_n^*\}. \tag{11}$$

It may be dangerous to use the rule give by the principle of optimality. In Example 2 of the introduction of this chapter, $V_n^* = 1$ for all finite n, and $Y_n < 1$ for all finite n so N^* tells you to continue forever. This gives a payoff of 0.

Using A1 only, N^* may not be optimal, but yet no rule is made worse by stopping when N^* tells you to stop, as the following lemma shows.

Lemma 4. *Assume A1. Let N be any stopping rule and let $N' = \min(N, N^*)$. Then, $\mathrm{E}Y_{N'} \geq \mathrm{E}Y_N$.*

Proof. On $\{N^* = n < N\}$, $V_n^* \geq \mathrm{E}(Y_N|\mathcal{F}_n)$. Hence,

$$\begin{aligned} \mathrm{E}I\{N^* < N\}Y_{N^*} &= \sum_0^\infty \mathrm{E}I\{N^* = n < N\}Y_n \\ &= \sum_0^\infty \mathrm{E}I\{N^* = n < N\}V_n^* \\ &\geq \sum_0^\infty \mathrm{E}(I\{N^* = n < N\}\mathrm{E}\{Y_N|\mathcal{F}_n\}) \\ &= \sum_0^\infty \mathrm{E}I\{N^* = n < N\}Y_N \\ &= \mathrm{E}I\{N^* < N\}Y_N. \end{aligned} \tag{12}$$

Hence,

$$\begin{aligned} \mathrm{E}Y_{N'} &= \mathrm{E}I\{N^* < N\}Y_{N*} + \mathrm{E}I\{N^* \geq N\}Y_N \\ &\geq \mathrm{E}I\{N^* < N\}Y_N + \mathrm{E}I\{N^* \geq N\}Y_N = \mathrm{E}Y_N, \end{aligned} \tag{13}$$

completing the proof. ∎

Under A1 and A2, N^* is optimal. In fact, the proof of the following theorem shows that out of all optimal rules it stops the soonest (for any optimal rule N^o, $N^* \leq N^o$). A characterization of all optimal rules may be found in the paper of M. Klass (1973).

Theorem 3. *Under A1, if there exists an optimal rule, in particular if A2 holds, then N^* is optimal.*

Proof. Let N_0 be an optimal rule and let $N = \min\{N_0, N^*\}$. Then from Lemma 4, N is also optimal and $N \leq N^*$. We will complete the proof by showing that $N = N^*$. Suppose $\mathrm{P}\{N < N^*\} > 0$. Then for some n, $\mathrm{P}\{N = n < N^*\} > 0$, and on $\{N = n < N^*\}$, $Y_n < V_n^*$, so that we should be able to improve N by changing N on $\{N = n < N^*\}$ to something that gives return close to V_n^*. As in the proof of Theorem 2, find N_k regular from n on such that $\sup_k \mathrm{E}\{Y_{N_k}|\mathcal{F}_n\} = V_n^*$ and let $N_k' = \max\{N_1, \ldots, N_k\}$ so that

$$V_n^* \geq \mathrm{E}(Y_{N_k'}|\mathcal{F}_n) \geq \max_{1\leq j\leq k} \mathrm{E}(Y_{N_j}|\mathcal{F}_n) \to V_n^* \tag{14}$$

monotonically. Then, since $\mathrm{E}I\{N = n < N^*\}Y_{N_k'} \to \mathrm{E}I\{N = n < N^*\}V_n^*$, there exists a k such that $\mathrm{E}I\{N = n < N^*\}Y_n < \mathrm{E}I\{N = n < N^*\}Y_{N_k'}$. Letting $N' = N_k'$ on $\{N = n < N^*\}$ and $N' = N$ otherwise, we have $\mathrm{E}Y_{N'} > \mathrm{E}Y_N$, contradicting the optimality of N. Consequently, $\mathrm{P}(N < N^*) = 0$ which implies that $N = N^*$. ■

Often useful in applications is an alternate form of the rule given by the principle of optimality, based on the random variables

$$W_n^* = \mathrm{E}(V_{n+1}^*|\mathcal{F}_n).$$

By the optimality equation, $V_n^* = \max(Y_n, W_n^*)$, and the rule, N^*, becomes

$$N^* = \min\{n \geq 0 : Y_n \geq W_n^*\}.$$

One can show that $W_n^* = \operatorname{ess\,sup}_{N>n}\mathrm{E}(Y_N|\mathcal{F}_n)$, so W_n^* can be considered as the best return available at stage n among rules that continue at least one stage. The rule N^* calls for stopping when the return for stopping is at least as great as the best that can be obtained by continuing. The rule $N^{**} = \min\{n \geq 0 : Y_n > W_n^*\}$ is the optimal rule that stops last (for any optimal rule N^o, $N^{**} \geq N^o$).

§**3.3 The Wald Equation.** The following equation, due to Wald, is very useful in solving optimal stopping problems.

Theorem 4. *Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables such that $\mathrm{E}|X_1| < \infty$, let $\mu = \mathrm{E}X_1$, and let $S_n = X_1 + \cdots + X_n$. Let N be any stopping rule, adapted to $X_1, X_2, \ldots$. Then, if $\mathrm{E}N < \infty$,*

$$\mathrm{E}(S_N) = \mu\mathrm{E}(N). \tag{15}$$

Moreover, if $\mathrm{E}N = \infty$ and $\mu \neq 0$, then (15) holds provided $\mathrm{E}S_N$ exists (that is, provided not both $\mathrm{E}S_N^+ = \infty$ and $\mathrm{E}S_N^- = \infty$ where $x^+ = \max(0, x)$ and $x^- = -\min(0, x)$).

We give two proofs. The first does not cover the "moreover" part of the statement, but is more flexible for providing simple extensions.

Proof #1. Provided the change of summation can be justified, we have

$$\begin{aligned}
\mathrm{E}S_N &= \sum_{n=1}^{\infty} \mathrm{E}\{\mathrm{I}(N = n)S_n\} \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{n} \mathrm{E}\{\mathrm{I}(N = n)X_j\} \\
&= \sum_{j=1}^{\infty}\sum_{n=j}^{\infty} \mathrm{E}\{\mathrm{I}(N = n)X_j\} \\
&= \sum_{j=1}^{\infty} \mathrm{E}\{\mathrm{I}(N \geq j)X_j\}.
\end{aligned} \tag{16}$$

Since N is a stopping rule for the sequence $X_1, X_2, \ldots$, the event $\{N \geq j\}$ depends only on $X_1, \ldots, X_{j-1}$ and so is independent of X_j. The equation continues

$$= \sum_{j=1}^{\infty} \mathrm{P}\{N \geq j\}\mathrm{E}(X_j) = \mu \sum_{j=1}^{\infty} \mathrm{P}\{N \geq j\} = \mu \mathrm{E}N.$$

The interchange of summations is justified provided the double summation in (16) converges absolutely. This follows by replacing X_j with $|X_j|$ in (16) and obtaining as above, $\mathrm{E}(|S_N|) \leq \mathrm{E}(|X_1|)\mathrm{E}(N) < \infty$.

Proof #2. (This proof is due to Blackwell (1946). The 'Moreover' part is due Robbins and Samuel (1966).) Consider n stopping problems as follows. Let N_1 be the stopping rule N applied to the sequence $X_1, X_2, \ldots$, let N_2 be the stopping rule N applied to $X_{N_1+1}, X_{N_1+2}, \ldots$, etc., and let N_n be the stopping rule N applied to $X_{N_1+\cdots+N_{n-1}+1}$, $X_{N_1+\cdots+N_{n-1}+2}$, ... Let the returns for these problems be denoted by $Z_1, \ldots, Z_n$ where $Z_j = X_{N_1+\cdots+N_{j-1}+1} + \cdots + X_{N_1+\cdots+N_j}$. Then, the Z_j are independent with the same distribution as S_N, and we have

$$\frac{Z_1 + \cdots + Z_n}{n} = \frac{X_1 + \cdots + X_{N_1+\cdots+N_n}}{N_1 + \cdots + N_n} \cdot \frac{N_1 + \cdots + N_n}{n}. \tag{17}$$

From the strong law of large numbers, $(X_1 + \cdots + X_{N_1+\cdots+N_n})/(N_1 + \cdots + N_n) \to \mu$ a.s. If $\mathrm{E}N < \infty$, then $(N_1 + \cdots + N_n)/n \to \mathrm{E}N$ a.s., so that $(Z_1 + \cdots + Z_n)/n \to \mu\mathrm{E}N$ a.s., and hence by the converse to the strong law, $\mathrm{E}S_N = \mu\mathrm{E}N$. Similarly, if $\mathrm{E}S_N < \infty$ and $\mu \neq 0$, then $(N_1 + \cdots + N_n)/n \to \mathrm{E}S_N/\mu = \mathrm{E}N$ a.s. ■

REMARK 1. One interpretation of this result for $\mu = 0$ is that it shows that no stopping strategy with $\mathrm{E}N < \infty$ in a sequence of identical fair games can yield a positive expected payoff. If each game has expectation $\mu = 0$, then the expectation of the sum S_N is zero for every stopping rule N with $\mathrm{E}N < \infty$. If $\mathrm{E}N = \infty$ is allowed, then we can obtain $\mathrm{E}S_N > 0$ by defining N as the first n such that $S_n > 0$, provided the distribution of X_1 is not degenerate at zero. The law of the iterated logarithm shows that N is finite with probability one.

REMARK 2. If the X_i are independent with mean μ but not identically distributed, then the result $\mathrm{E}S_N = \mu\mathrm{E}N$ may not hold. (See Exercise 4.) The natural generalization of this theorem to the dependent case is for sequences S_n that form a martingale. For this theory, see for example the books of Chung (1968), or Chow, Robbins and Siegmund (1971).

REMARK 3. If $\mathrm{E}X_1^- < \infty$, the argument involving (16) shows that when $\mathrm{E}N < \infty$, $\mathrm{E}S_N = \mathrm{E}X_1\mathrm{E}N$ even if $\mathrm{E}X_1 = \infty$, provided (when $N \equiv 0$) 0 times ∞ is taken to be 0.

EXAMPLE 3. As an example of the use of this equation, suppose that the X_i are i.i.d. taking integer values less than or equal to 1 with probabilities $\mathrm{P}(X_i = j) = p_j$, where $\sum_{-\infty}^{1} p_j = 1$, and assume that $\mathrm{E}X_i > 0$. For some integer $r > 0$, let $N = \min\{n > 0 : S_n = r\}$. Then $N < \infty$ with probability one, since $S_n \to \infty$ a.s. by the law of large numbers. Then, since $S_N \equiv r$, we have $\mathrm{E}N = \mathrm{E}S_N/\mathrm{E}X_1 = r/\mathrm{E}X_1$. In the particular case where $p_1 = p$ and $p_0 = 1-p$, we have the well-known result that the expectation of the number of trials until the rth success in a sequence of i.i.d. Bernoulli trials is r/p (a negative binomial random variable).

§**3.4 Prophet Inequalities.** Under A1 and A2, the decision maker can attain an expected payoff of $V^* = \sup_N \mathrm{E}Y_N$, using stopping rules, i.e. using rules for which stopping at stage n depends only on the observations up to that time. On the other hand, a prophet, who can foresee the future and who knows the values of all the Y_n, would stop at some Y_n which is close to the largest overall, and thus achieve an expected payoff as close as desired to $M^* = \mathrm{E}\sup_n Y_n$. We certainly have $V^* \le M^*$.

Obviously, the prophet has a big advantage over the decision maker. Therefore it is surprising that in many situations there are universal upper bounds on the advantage of the prophet over the decision maker. Inequalities bounding M^* above by some function of V^* are called prophet inequalities. The most basic and surprising of these is the first general prophet inequality discovered. It is due to Krengel, Sucheston and Garling, and found in Krengel and Sucheston (1977, 1978).

This inequality deals with a sequence of random variables $X_1, X_2, \ldots$ that are *independent* and *nonnegative*, and in which the payoff, Y_n, for stopping at stage n is X_n itself. We require that the decision maker use rules that stop at some finite time (i.e. we require $\mathrm{P}(N < \infty) = 1$), so there is no need to define Y_∞. We use the notation,

$$V^* = \sup_{N<\infty} \mathrm{E}(X_N) \qquad \text{and} \qquad M^* = \mathrm{E}(\sup_n X_n). \tag{18}$$

The basic theorem of Krengel, Sucheston and Garling is the following: *For independent nonnegative random variables,* $X_1, X_2, \ldots,$

$$M^* \le 2V^*. \tag{19}$$

Thus, a prophet cannot win more on the average than twice that of a real-time decision maker, when dealing with a sequence of independent nonnegative random variables.

This inequality follows from the finite horizon version, where the decision maker is restricted to using rules that stop by some prespecified time n. In this case we have

$$V_n^* = \sup_{N\le n} \mathrm{E}(X_N) \qquad \text{and} \qquad M_n^* = \mathrm{E}(\max_{1\le j\le n} X_j). \tag{20}$$

and the inequality becomes

$$M_n^* \le 2V_n^*. \tag{21}$$

To see that (21) implies (19), simply note, since $\max_{1\le j\le n} X_j$ is a.s. nondecreasing to $\sup_n X_n$, that $M_n^* \to M^*$ as $n \to \infty$. Then $V^* \ge V_n^* \ge (1/2)M_n^* \to (1/2)M^*$ as $n \to \infty$.

Examples. One cannot replace the condition that the X_i be nonnegative with the condition that the expectations be nonnegative. Here is an example with $n = 2$. Let the $X_1 = 1$ and let X_2 be B with probability $1/2$ and $-B$ with probability $1/2$, for some large number B. The best one can do with stopping rules is to stop with the first observation and receive $V_2^* = 1$, since continuing gives an expected value of 0. But the prophet can get $M_2^* = \mathrm{E}\max\{1, X_2\} = (1/2) + (1/2)B = (B+1)/2$.

A similar example shows that the inequalities (19) and (21) are sharp. Let $X_1 = 1$, $X_3 = X_4 = \cdots = 0$, and for some number $B > 1$ let $\mathrm{P}(X_2 = B) = 1/B$ and $\mathrm{P}(X_2 = 0) = 1 - (1/B)$. Then again the best the real-time decision maker can do is $V^* = 1$. Whereas the prophet can obtain $M^* = \mathrm{E}\max\{1, X_2\} = (1 - (1/B)) + (1/B)B = 2 - (1/B)$. This can be made as close to 2 as desired by making B large.

If in (21) we rule out the trivial cases where $M^* = 0$ and $M^* = \infty$, then the inequality may be taken to be strict.

The inequality (21) is interesting because of its generality. Of course, in particular situations the bound may not be very good. For example, in the Cayley-Moser problem of Chapter 2, when $X_1, \ldots, X_n$ are i.i.d. uniform, the distribution of $\max\{X_1, \ldots, X_n\}$ has the beta distribution function, $F(y) = y^n$ on $[0,1]$. This distribution has mean $M_n^* = n/(n+1) = 1 - (1/(n+1))$. This is just slightly bigger than $V_n^* = A_n \simeq 1 - (2/(n + \log(n) + 1.768))$ when n is large. ■

For the proof of the prophet inequality (21), we use a method due to Ester Samuel-Cahn (1984). This proof has the advantage of being constructive. It exhibits a stopping rule that achieves at least half of the profit of a prophet. Moreover, it shows that the advantage of the prophet does not increase if the decision maker is restricted to using pure threshold rules. (A threshold rule is pure if the cutoff point does not depend on n.)

Let $M_n = \max_{1 \le j \le n} X_j$. We define a **pure threshold stopping rule** with threshold c to be a rule of the form

$$s(c) = \begin{cases} \min\{1 \le j \le n : X_j > c\} & \text{if } M_n > c \\ n & \text{if } M_n \le c \end{cases} \tag{22}$$

or

$$t(c) = \begin{cases} \min\{1 \le j \le n : X_j \ge c\} & \text{if } M_n \ge c \\ n & \text{if } M_n < c. \end{cases} \tag{23}$$

Let m denote a median of the distribution of M_n, i.e.

$$\mathrm{P}(M_n < m) = q \le 1/2 \qquad \text{and} \qquad \mathrm{P}(M_n > m) = p \le 1/2. \tag{24}$$

and let

$$\beta = \sum_1^n \mathrm{E}(X_i - m)^+. \tag{25}$$

Then one of the two stopping rules, $s(m)$ or $t(m)$, will achieve at least half of $\mathrm{E}M_n$. More precisely,

Theorem 5. *If* $m \le \beta$, *then* $\mathrm{E}(M_n) \le 2\mathrm{E}(X_{s(m)})$. *If* $m \ge \beta$, *then* $\mathrm{E}(M_n) \le 2\mathrm{E}(X_{t(m)})$.

Proof. First note that $(M_n - m)^+ = \max_{1\le i\le n}(X_i - m)^+ \le \sum_1^n (X_i - m)^+$ a.s., so that $\mathrm{E}(M_n - m)^+ \le \beta$. Hence,

$$\mathrm{E}(M_n) = m + \mathrm{E}(M_n - m) \le m + \mathrm{E}(M_n - m)^+ \le m + \beta. \tag{26}$$

Suppose first that $m \le \beta$. Then

$$\begin{aligned} \mathrm{E}(X_{s(m)} - m)^+ &= \mathrm{E}\sum_1^n (X_i - m)^+ \mathrm{I}(s(m) = i) \\ &= \sum_1^n \mathrm{E}(X_i - m)^+ \mathrm{I}(s(m) > i-1) \\ &= \sum_1^n \mathrm{E}(X_i - m)^+ \mathrm{P}(s(m) > i-1). \end{aligned} \tag{27}$$

The last equality uses the fact that the event $\{s(m) > i-1\}$ depends only on $X_1, \ldots, X_{i-1}$ and so is independent of X_i. Moreover,

$$\mathrm{P}(s(m) > i-1) \ge \mathrm{P}(s(m) = n) \ge \mathrm{P}(M_n \le m) = 1 - p \tag{28}$$

for all i. Together (27) and (28) show that $\mathrm{E}(X_{s(m)} - m)^+ \ge \beta(1-p)$. Finally,

$$\mathrm{E}(X_{s(m)}) = \mathrm{E}(X_{s(m)}\mathrm{I}(X_{s(m)} > m)) + \mathrm{E}(X_n \mathrm{I}(X_{s(m)} \le m) \ge \mathrm{E}(X_{s(m)}\mathrm{I}(X_{s(m)} > m)) \tag{29}$$

since $X_n \ge 0$. Continuing this inequality,

$$\begin{aligned} \mathrm{E}(X_{s(m)}\mathrm{I}(X_{s(m)} > m)) &= \mathrm{E}((X_{s(m)} - m)\mathrm{I}(X_{s(m)} > m)) + m\mathrm{P}(X_{s(m)} > m) \\ &= \mathrm{E}(X_{s(m)} - m)^+ + m\mathrm{P}(M_n > m) = \mathrm{E}(X_{s(m)} - m)^+ + mp \\ &\ge \beta(1-p) + mp = \beta - (\beta - m)p \\ &\ge \beta - (\beta - m)/2 = (m+\beta)/2 \\ &\ge \mathrm{E}(M_n)/2 \end{aligned} \tag{30}$$

using (26) and our assumption that $m \le \beta$. This proves the first statement of the theorem. The second statement is proved in a completely analogous fashion. ∎

It is interesting to note that this proof shows that for inequality (21) the assumption that all the X_i be nonnegative may be replaced by the assumption that just X_n be nonnegative. For the corresponding improvement in the conditions for inequality (19), see Exercise 6.

There are many other prophet inequalities. An important class of prophet inequalities concerns a sequence of independent uniformly bounded random variables. The basic theorem in this class is due to Hill and Kertz (1981): *For independent random variables,* $X_1, X_2, \ldots$ *such that* $0 \le X_i \le 1$ *for all* i,

$$M^* \le V^* + \frac{1}{4} \tag{31}$$

or better, as proved in Hill (1983)

$$M^* \le 2V^* - (V^*)^2. \tag{32}$$

The inequality here is sharp and may be attained. (See Exercise 7.)

Other classes of prophet inequalities include those where the variables are allowed to be dependent, or the variables are restricted to being i.i.d., or where the decision maker is given the freedom to choose the order in which the X_i are observed. For a review of these and other prophet inequalities, see the survey paper of Hill and Kertz (1992).

§3.5 Exercises.

1. **The one-stage look-ahead rule.** Let N_1 denote the one-stage look-ahead rule, sometimes called **the myopic rule**,

$$N_1 = \min\{n \ge 0 : Y_n \ge \mathrm{E}(Y_{n+1}|\mathcal{F}_n)\},$$

and let $N_1^{(J)}$ denote this rule truncated at J, $N_1^{(J)} = \min\{N_1, J\}$.
(a) Show that $N_1^{(J)}$ is regular if A1 is satisfied.
(b) Show that N_1 is regular if A1 and A2 are satisfied.
(c) Show by counterexample that if A1 or A2 is not satisfied, N_1 may not be regular.

2. **The hypermetropic rule.** Let N_∞ denote the hypermetropic rule,

$$N_\infty = \min\{n \ge 0 : Y_n \ge \mathrm{E}(Y_\infty|\mathcal{F}_n)\}.$$

Show that under condition A1, N_∞ is regular.

3. **The one-time look-ahead rule.** Let T_1 denote the one-time look-ahead rule,

$$T_1 = \min\{n \ge 0 : Y_n \ge \sup_{j>n, j\le\infty} \mathrm{E}(Y_j|\mathcal{F}_n)\}.$$

(a) Show that T_1 is regular under conditions A1 and A2.
(b) Assuming A1 and A2, conclude that the one-time look-ahead rule is at least as good as the one-stage look-ahead rule, and the hypermetropic rule.

4. **Wald's equation without the assumption of identical distributions.**
(a) Find an example of independent $X_1, X_2, \ldots$ such that $\mathrm{E}X_j = 0$ for all j, and a stopping rule N adapted to $X_1, X_2, \ldots$ with $\mathrm{E}N < \infty$ such that $\mathrm{E}(X_1 + \cdots + X_N) > 0$.
(b) Let $X_1, X_2, \ldots$ be independent with finite means $\mu_j = \mathrm{E}X_j$ for $j = 1, 2, \ldots$. Let N

be any stopping rule such that $\mathrm{E}N < \infty$. Find the extra conditions on the distributions of the X_j that are needed so that equation (16) goes through to show

$$\mathrm{E}(S_N) = \mathrm{E}\sum_{j=1}^{N} \mu_j. \tag{33}$$

5. Give an example of independent nonnegative random variables, $X_1, \ldots, X_n$, for which $\mathrm{E}(M_n) > 2\mathrm{E}(X_{s(m)})$, and another example for which $\mathrm{E}(M_n) > 2\mathrm{E}(X_{t(m)})$. (For this, it suffices to take $n = 2$.)

6. Prove the following extension of basic prophet inequality: If $X_1, X_2, \ldots$, are independent, and if $\sum_{i=1}^{\infty} \mathrm{P}(X_i \geq 0) = \infty$, then (19) holds.

7. For each value of V^* in (0,1), give an example of independent random variables $X_1, X_2, \ldots$ with $0 \leq X_i \leq 1$ for all i such that $M^* = 2V^* + (V^*)^2$.