

Final Examination
Statistics 200C

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1. (a) Define: X_n converges in probability to X .
(b) Define: X_n converges in quadratic mean to X .
(c) Show that if X_n converges in quadratic mean to X , then X_n converges in probability to X .

2. Let X_1, X_2, \dots be independent random variables with X_k having a uniform distribution over the interval $(0, k)$. Let $S_n = \sum_{k=1}^n X_k$.
(a) Find $E(S_n)$ and $\text{Var}(S_n)$.
(b) Show that $(S_n - E(S_n))/\sqrt{\text{Var}(S_n)}$ converges in law to the standard normal distribution by checking the Lindeberg condition.

3. Let X_1, \dots, X_n be a sequence of i.i.d. random variables taking values in the set {red,white,blue} with $P(X_j = \text{red}) = P(X_j = \text{white}) = P(X_j = \text{blue}) = 1/3$. Let S_n denote the total number of times that the sequence (red,white,blue) appears in that order from left to right in the sequence. That is, if $Z_j = I(X_j = \text{red}, X_{j+1} = \text{white}, X_{j+2} = \text{blue})$, then $S_n = \sum_{j=1}^{n-2} Z_j$.
(a) Find $E(S_n)$ and $\text{Var}(S_n)$.
(b) What is the limiting distribution of $(S_n - ES_N)/\sqrt{n}$?

4. Let X_1, \dots, X_n be a sample from the Pareto distribution with distribution function $F(x) = (1 - (1/x))I(x > 1)$ and density $f(x) = (1/x^2)I(x > 1)$.
(a) Find the first quartile and the median of the distribution.
(b) Find the joint asymptotic distribution of the sample first quartile and the sample median.
(c) Let M_n denote the sample maximum. Find the limiting distribution function of M_n/b_n for suitable choice of b_n to get a non-degenerate limit.

5. Let X_1, \dots, X_n be a sample from the logistic distribution with density

$$f(x|\theta) = \frac{e^{-(x-\theta)}}{(1 + e^{-(x-\theta)})^2}$$

- (a) Find the likelihood equation for solving for the maximum likelihood estimate.
- (b) This density is symmetric about θ , so θ is the median. What is the asymptotic distribution of the sample median as a estimate of θ ?
- (c) Show how to improve the median to a fully efficient estimate using the likelihood equation. (Note: Fisher information is $\mathcal{I}(\theta) = 1/3$.)

6. (a) Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a sample from a bivariate distribution with mean (μ_x, μ_y) , and covariance matrix, $\mathfrak{X} = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}$. As an estimate of $\theta = \mu_x/\mu_y$, one may use simply $\theta_n^* = \bar{X}_n/\bar{Y}_n$. Assume $\mu_y \neq 0$, and find the asymptotic distribution of θ_n^* .

(b) Specialize to the case where the distribution of Y_i is exponential with mean 1, and where the conditional distribution of X_i given $Y_i = y_i$ is normal with mean, θy_i , and variance, 1. First note that $E(X_i) = \theta$ and $E(Y_i) = 1$, so that $\mu_x/\mu_y = \theta$. Find \mathfrak{X} and the asymptotic distribution of θ_n^* in this case.

(c) In case (b), find the maximum likelihood estimate of θ . What is the asymptotic efficiency of θ_n^* found in (b).

7. Let X_1, \dots, X_n be a sample from the exponential distribution with mean θ for some $\theta > 0$, (density $f(x|\theta) = \theta^{-1}e^{-x/\theta}\mathbf{I}(x > 0)$), and let Y_1, \dots, Y_n be an independent sample from $f(y|\mu)$ for some $\mu > 0$.

(a) Find the likelihood ratio test statistic for testing the hypothesis $H_0 : \mu = 2\theta$ based on X_1, \dots, X_n , and Y_1, \dots, Y_n .

(b) What is its asymptotic distribution for large n under H_0 ?

8. A sample of size n is taken in a multinomial experiment with c^2 cells denoted (i, j) , $i = 1, \dots, c$ and $j = 1, \dots, c$. Let p_{ij} denote the probability of cell (i, j) , and let n_{ij} denote the number falling in cell (i, j) , so that $\sum \sum p_{ij} = 1$ and $\sum \sum n_{ij} = n$.

(a) Let H denote the hypothesis of symmetry, that $p_{ij} = p_{ji}$ for all i and j . Find the chi-square test of H against all alternatives? How many degrees of freedom does it have?

(b) Let H_0 denote the hypothesis of symmetry and independence, namely that $p_{ij} = \pi_i\pi_j$ for some probability vector, (π_1, \dots, π_c) . Find the chi-square test of H_0 against all alternatives. How many degrees of freedom?

(c) What, then, is the chi-square test of H_0 against H , and how many degrees of freedom does it have?

Solutions to the Final Examination, Stat 200C, Spring 2009.

1. (a) X_n converges to X in probability if for all $\epsilon > 0$, $P(|X_n - X| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$.

(b) X_n converges to X in quadratic mean if $E(|X_n - X|^2) \rightarrow 0$ as $n \rightarrow \infty$.

(c) This is done using a Chebychev argument. Let $\epsilon > 0$. Since

$$E(X_n - X)^2 \geq E\{(X_n - X)^2 I(|X_n - X| > \epsilon)\} \geq \epsilon^2 P(|X_n - X| > \epsilon),$$

we see that if $E(|X_n - X|^2) \rightarrow 0$, then $P(|X_n - X| > \epsilon) \rightarrow 0$.

$$2. \text{ (a) } E(S_n) = \sum_{k=1}^n E(X_k) = \sum_{k=1}^n (k/2) = n(n+1)/4.$$

$$B_n^2 = \text{Var}(S_n) = \sum_{k=1}^n \text{Var}(X_k) = \sum_{k=1}^n (k^2/12) = n(n+1)(2n+1)/72.$$

(b) Let $Z_k = X_k - (k/2)$ so that $EZ_k = 0$ and $Z_k < k/2$ a.s. Then

$$\begin{aligned} \frac{1}{B_n^2} \sum_{k=1}^n E\{Z_k^2 I(Z_k^2 > \epsilon^2 B_n^2)\} &\geq \frac{1}{B_n^2} \sum_{k=1}^n E\{Z_k^2 I((k/2)^2 > \epsilon^2 B_n^2)\} \\ &\geq \frac{1}{B_n^2} \sum_{k=1}^n E\{Z_k^2 I((n/2)^2 > \epsilon^2 B_n^2)\} = I((n/2)^2 > \epsilon^2 B_n^2) \end{aligned}$$

This goes to zero as $n \rightarrow \infty$ since B_n goes to infinity at rate n^3 .

3. (a) The Z_j form a 2-dependent stationary sequence. $E(S_n) = \sum_{j=1}^{n-2} E(Z_j) = (n-2)E(Z_1) = (n-2)/27$. $\text{Var}(S_n) = (n-2)\text{Var}(Z_1) + 2(n-3)\text{Cov}(Z_1, Z_2) + 2(n-4)\text{Cov}(Z_1, Z_3) = (n-2)(26/27^2) - 2(n-3)/27^2 - 2(n-4)/27^2$.

(b) $(S_n - ES_n)/\sqrt{n} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 22/27^2)$.

4. (a) The first quartile is $x_{.25} = 4/3$ and the median is $x_{.5} = 2$.

(b) $f(x_{.25}) = f(4/3) = (3/4)^2$, and $f(x_{.5}) = f(2) = (1/2)^2$. So

$$\sqrt{n} \begin{pmatrix} X_{\lceil n/4 \rceil} - 4/3 \\ X_{\lceil n/2 \rceil} - 2 \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 16/27 & 8/9 \\ 8/9 & 4 \end{pmatrix}\right)$$

(c) $P(M_n/b_n \leq x) = P(M_n \leq b_n x) = F(b_n x)^n = (1 - (1/(b_n x)))^n I(x > 1/b_n) \rightarrow \lim_{n \rightarrow \infty} \exp\{-(n/b_n x)\} I(x > 1/b_n) = \exp\{-1/x\} I(x > 0)$, if we take $b_n = n$.

5. (a) The log likelihood function is

$$\ell_n(\theta) = - \sum_{j=1}^n (X_j - \theta) - 2 \sum_{j=1}^n \log(1 + \exp\{-(X_j - \theta)\})$$

The likelihood equation is $\dot{\ell}_n = 0$, or

$$n - 2 \sum_{j=1}^n \frac{1}{\exp\{X_j - \theta\} + 1} = 0.$$

(b) Let $\tilde{\theta}_n$ denote the sample median. Then, since $f(\theta|\theta) = 1/4$, $\sqrt{n}(\tilde{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, (1/4)/f(\theta, \theta)^2) = \mathcal{N}(0, 4)$.

(c) We may use $\hat{\theta}_n = \tilde{\theta}_n - \ddot{\ell}_n(\tilde{\theta}_n)^{-1} \dot{\ell}_n(\tilde{\theta}_n)$ or more simply

$$\hat{\theta}_n = \tilde{\theta}_n + \mathcal{I}^{-1}(\tilde{\theta}_n) \frac{1}{n} \dot{\ell}_n(\tilde{\theta}_n) = \tilde{\theta}_n + 3 \left[1 - \frac{2}{n} \sum_1^n \frac{1}{\exp\{X_j - \tilde{\theta}_n\} + 1} \right].$$

6. (a) By the Central Limit Theorem, $\sqrt{n}((\bar{X}_n, \bar{Y}_n) - (\mu_x, \mu_y)) \xrightarrow{\mathcal{L}} \mathcal{N}((0, 0), \mathbb{V})$. Then we apply Cramér's Theorem with $g(x, y) = x/y$ and $\dot{g}(x, y) = (1/y, -x/y^2)$, so that $\dot{g}(\mu_x, \mu_y) = (1/\mu_y)(1, -\theta)$. We find

$$\begin{aligned} \sqrt{n}(\theta_n^* - \theta) &\xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{\mu_y^2}(1, -\theta) \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \begin{pmatrix} 1 \\ -\theta \end{pmatrix}\right) \\ &= \mathcal{N}\left(0, \frac{1}{\mu_y^2}(\sigma_x^2 - 2\theta\sigma_{xy} + \theta^2\sigma_y^2)\right). \end{aligned}$$

(b) For the exponential distribution of Y , $\mu_y = 1$ and $\sigma_y^2 = 1$. Then $E(X) = E(E(X|Y)) = E(\theta Y) = \theta$. Similarly, $E(X^2) = E(E(X^2|Y)) = E(1 + \theta^2 Y^2) = 1 + 2\theta^2$, $\sigma_x^2 = 1 + \theta^2$, $E(XY) = E(E(XY|Y)) = E(YE(X|Y)) = E(\theta Y^2) = 2\theta$, and $\sigma_{xy} = \theta$. So in this case, $\sqrt{n}(\theta_n^* - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$.

(c) In case (b), the joint density of (X, Y) is $f(x, y|\theta) = (2\pi)^{-1/2} e^{-y} e^{-(x-\theta y)^2/2}$ for $y > 0$. We find

$$(\partial/\partial\theta) \log f(x, y|\theta) = (x - \theta y)y.$$

From this we can see that the maximum likelihood estimate of θ is $\hat{\theta}_n = (\sum X_i Y_i / \sum Y_i^2)$. From $(\partial/\partial\theta)^2 f(x, y|\theta) = -y^2$, we see that Fisher information is $\mathcal{I}(\theta) = E(Y^2) = 2$. Therefore, $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1/2)$. The asymptotic efficiency of θ_n^* (relative to $\hat{\theta}_n$) is only 50%.

7. (a) $L_n(\theta, \mu) = \theta^{-n} \mu^{-n} \exp\{-(1/\theta) \sum_1^n X_i - (1/\mu) \sum_1^n Y_i\}$. The maximum likelihood estimates under the general hypothesis are $\hat{\theta}_n = \bar{X}_n$ and $\hat{\mu}_n = \bar{Y}_n$. Under H_0 , they are $\tilde{\theta}_n = (1/2)\bar{X}_n + (1/4)\bar{Y}_n$ and $\tilde{\mu}_n = 2\tilde{\theta}_n$. the likelihood ratio test statistic is

$$\Lambda_n = \frac{L_n(\tilde{\theta}_n, \tilde{\mu}_n)}{L_n(\hat{\theta}_n, \hat{\mu}_n)} = \frac{2^{-n} \tilde{\theta}_n^{-2n} e^{-2n}}{\hat{\theta}_n^{-n} \hat{\mu}_n^{-n} e^{-2n}} = \left[\frac{\bar{X}_n \bar{Y}_n}{2((1/2)\bar{X}_n + (1/4)\bar{Y}_n)^2} \right]^n$$

(b) Under H_0 , $-2 \log(\Lambda_n)$ converges in law to a chisquare distribution with 1 degree of freedom.

8. (a) Under H , the maximum likelihood estimates of the p_{ij} are $\hat{p}_{ii} = n_{ii}/n$ and for $i \neq j$, $\hat{p}_{ij} = (n_{ij} + n_{ji})/2n$. There are $c + (c-1) + (c-2) + \dots + 1 = c(c+1)/2$ restrictions going from the general hypothesis to H . So the chisquare test of H rejects H if $\chi^2(\hat{p})$

is greater than the appropriate cutoff point for a chisquare distribution with $c(c - 1)/2$ degrees of freedom.

(b) Under H_0 , the likelihood is proportional to $\prod_i \prod_j (\pi_i \pi_j)^{n_{ij}} = (\prod_i \pi_i^{n_{i.}})(\prod_j \pi_j^{n_{.j}}) = \prod_i \pi_i^{n_{i.} + n_{.i}}$. So the maximum likelihood estimates are $\tilde{\pi}_i = (n_{i.} + n_{.i})/(2n)$. There are $c - 1$ parameters estimated so the chisquare test of H_0 rejects H_0 if $\chi^2(\tilde{\mathbf{p}})$ is greater than the appropriate cutoff point for a chisquare distribution with $(c^2 - 1) - (c - 1) = c(c - 1)$ degrees of freedom.

(c) The chisquare test of H_0 within H , rejects H_0 if $\chi^2(\tilde{\mathbf{p}}) - \chi^2(\hat{\mathbf{p}})$ is greater than the appropriate cutoff point for a chisquare distribution with $c(c - 1) - (c(c - 1)/2) = c(c - 1)/2$ degrees of freedom.