Solutions to Exercise Set 9.

20.5. (a) $E_\theta(X) = (1 - \theta) + 2\theta = 1 + \theta$, so the method of moments estimator of $\theta$ is $\hat{\theta}_n = \bar{X}_n - 1$.

(b) The log likelihood is $\ell_n(\theta) = \sum \log[(1 - \theta)e^{-x_i} + \theta(1/2)e^{-x_i/2}]$, so the likelihood equation is

$$\ell'_n(\theta) = \sum \frac{-e^{-x_i/2} + (1/2)}{(1 - \theta)e^{-x_i/2} + \theta(1/2)} = 0.$$  

One may improve $\hat{\theta}_n$ by the Newton method formula $\hat{\theta}_n = \hat{\theta}_n - \ell'_n(\hat{\theta}_n)/\ell''_n(\hat{\theta}_n)$, where

$$\ell''_n(\theta) = -\sum \frac{(e^{-x_i/2} - (1/2))^2}{[(1 - \theta)e^{-x_i/2} + \theta(1/2)]^2}.$$

22.1. (a) The likelihood function is $L = \prod (1/\sqrt{2\pi}\sigma)e^{-(y_i - \alpha - \beta x_i)^2/2\sigma^2}$. The MLE's under the general hypothesis, $H$, are $\tilde{\beta} = s_{xy}/s_x^2$, $\tilde{\alpha} = Y - \beta \overline{X}$, and $\tilde{\sigma}^2 = \frac{1}{n} \sum (Y_i - \tilde{\alpha} - \tilde{\beta} x_i)^2$. We have $\sup_H L = (1/\sqrt{2\pi}\sigma)^n e^{-n/2}$. Under $H_0$, the MLE's are $\hat{\alpha} = \hat{\beta} = \frac{1}{n} \sum Y_i (x_i + 1)/\sum (x_i + 1)^2$, and $\hat{\sigma}^2 = \frac{1}{n} \sum (Y_i - \hat{\alpha} (x_i + 1))^2$. We have $\sup_{H_0} L = (1/\sqrt{2\pi}\sigma)^n e^{-n/2}$. The likelihood ratio statistic is $\Lambda = (\hat{\sigma}/\tilde{\sigma})^n$. The likelihood ratio test rejects $H_0$ if $\Lambda$ is too small. This is equivalent to the test that rejects $H_0$ if $\tilde{\sigma}^2/\hat{\sigma}^2$ is too large, or if $F = (\tilde{\sigma}^2/\hat{\sigma}^2)/(\frac{1}{n-2} \hat{\sigma}^2)$ is too large. This is the usual $F$-test of $H_0$. Under $H_0$, the statistic $F$ has an $F$-distribution with 1 and $n - 2$ degrees of freedom.

(b) If $\sigma^2$ is given, the estimates of $\alpha$ and $\beta$ are the same as in (a). Here we have $\sup_H L = (\frac{1}{\sqrt{2\pi}\sigma})^n e^{-n\tilde{\sigma}^2/2\sigma^2}$, and $\sup_{H_0} L = (\sqrt{2\pi}\sigma)^n e^{-n\hat{\sigma}^2/2\sigma^2}$. The likelihood ratio statistic is $\Lambda = \exp\{-\frac{n}{2\sigma^2}(\tilde{\sigma}^2 - \hat{\sigma}^2)\}$. The likelihood ratio test rejects $H_0$ if $\Lambda$ is too small. This is equivalent to the test that rejects $H_0$ if $\tilde{\sigma}^2 - \hat{\sigma}^2$ is too large. This statistic is the numerator of the $F$ above. It has a chi-square distribution with 1 degree of freedom.

22.5. (a) The likelihood function is $L(\mu, \theta) = e^{-n\mu} \mu^{\Sigma X_i} \theta^{\Sigma Y_i}/(\prod x_i! y_i!)$. The maximum likelihood estimates under the general hypothesis are $\hat{\mu}_n = \overline{X}_n$ and $\hat{\theta}_n = \overline{Y}_n$. To find the maximum likelihood estimates under $H_0$, we replace $\mu$ by $\theta^2$ in log $L$ and take a derivative with respect to $\theta$.

$$\frac{\partial}{\partial \theta} \log L_n = -2n\theta + 2 \frac{\sum X_i}{\theta} - n + \frac{\sum Y_i}{\theta} = 0$$

Solving the quadratic for $\theta$ gives $\hat{\theta}_n = (-1 + \sqrt{1 + 8(2\bar{X} + \bar{Y})})/4$ as the MLE of $\theta$ under $H_0$. The likelihood ratio statistic then becomes

$$\lambda_n = \frac{L(\tilde{\theta}_n, \hat{\theta}_n)}{L(\hat{\mu}_n, \hat{\theta}_n)} = \frac{e^{-n\tilde{\theta}_n^2} - n\tilde{\theta}_n \tilde{\theta}_n \Sigma X + n\Sigma Y}{e^{-n\mu} - n\tilde{\theta}_n \mu \Sigma X + n\Sigma Y}$$

(b) $-2 \log \lambda_n \overset{L}{\longrightarrow} \chi^2_1$ as $n \to \infty$. 