17.4. (a)

\[ p_n = P(H_1|X_1, \ldots, X_n) = \frac{P(H_1, X_1, \ldots, X_n)}{P(X_1, \ldots, X_n)} = \frac{p_0 \prod_{i=1}^{n} f_0(X_i)}{p_0 \prod_{i=1}^{n} f_1(X_i) + (1 - p_0) \prod_{i=1}^{n} f_0(X_i)} . \]

(b) Assuming \( H_0 \) is true, we are to show

\[ -\frac{1}{n} \log(p_n) = \frac{1}{n} \log \left( 1 + \frac{1 - p_0}{p_0} \prod_{i=1}^{n} \frac{f_0(X_i)}{f_1(X_i)} \right) \xrightarrow{a.s.} K(f_0, f_1). \]

Since \( \frac{1}{n} \sum_1^n \log(f_0(X_i)/f_1(X_i)) \xrightarrow{a.s.} K(f_0, f_1) > 0 \), we have \( \prod_1^n (f_0(X_i)/f_1(X_i)) = \exp\{\sum_1^n \log(f_0(X_i)/f_1(X_i))\} = \exp\{n(\frac{1}{n} \sum_1^n \log(f_0(X_i)/f_1(X_i))\} \xrightarrow{a.s.} +\infty \). Therefore,

\[ \frac{1}{n} \log \left( 1 + \frac{1 - p_0}{p_0} \prod_{i=1}^{n} \frac{f_0(X_i)}{f_1(X_i)} \right) \sim \frac{1}{n} \log \left( \frac{1 - p_0}{p_0} \prod_{i=1}^{n} \frac{f_0(X_i)}{f_1(X_i)} \right) = \frac{1}{n} \log \left( \frac{1 - p_0}{p_0} \right) + \frac{1}{n} \sum_{i=1}^{n} \log \left( \frac{f_0(X_i)}{f_1(X_i)} \right) \xrightarrow{a.s.} K(f_0, f_1). \]

18.6. (a) By the Central Limit Theorem, \( \sqrt{n}(\bar{X}_n, \bar{Y}_n - (\mu_x, \mu_y)) \xrightarrow{\mathcal{L}} \mathcal{N}((0, 0), \mathcal{I}) \).

Then we apply Cramér’s Theorem with \( g(x, y) = x/y \) and \( \dot{g}(x, y) = (1/y, -x/y^2) \), so that \( \dot{g}(\mu_x, \mu_y) = (1/\mu_y)(1, -\theta) \). We find

\[ \sqrt{n}(\theta_n^* - \theta) \xrightarrow{\mathbb{P}} \mathcal{N}(0, \frac{1}{\mu_y^2}(1, -\theta) \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \begin{pmatrix} 1 \\ -\theta \end{pmatrix}) = \mathcal{N}(0, \frac{1}{\mu_y^2}(\sigma_x^2 - 2\theta\sigma_{xy} + \theta^2\sigma_y^2)). \]

(b) For the exponential distribution of \( Y \), \( \mu_y = 1 \) and \( \sigma_y^2 = 1 \). Then \( \text{E}(X) = \text{E}((E(X|Y)) = \text{E}(\theta Y) = \theta \).

Similarly, \( \text{E}(X^2) = \text{E}((E(X^2|Y)) = \text{E}(1 + \theta^2 Y^2) = 1 + 2\theta^2, \sigma_x^2 = 1 + \theta^2, \text{E}(XY) = \text{E}((E(XY|Y)) = \text{E}(\theta Y^2) = 2\theta, \text{and } \sigma_{xy} = \theta \).

So in this case, \( \sqrt{n}(\theta_n^* - \theta) \xrightarrow{\mathbb{P}} \mathcal{N}(0, 1) \).

(c) In case (b), the joint density of \( (X, Y) \) is \( f(x, y|\theta) = (2\pi)^{-1/2} e^{-y-\theta y^2/2} \) for \( y > 0 \). We find

\( (\partial/\partial \theta) \log f(x, y|\theta) = (x - \theta y)y \).
From this we can see that the maximum likelihood estimate of \( \theta \) is \( \hat{\theta}_n = \left( \sum X_i Y_i / \sum Y_i^2 \right) \).

From \( (\partial/\partial \theta)^2 f(x, y | \theta) = -y^2 \), we see that Fisher information is \( I(\theta) = E(Y^2) = 2 \).

Therefore, \( \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} N(0, 1/2) \). The asymptotic efficiency of \( \theta^*_n \) relative to \( \hat{\theta}_n \) is only 50%.

19.3. (a) The density is \( f(x | \mu, \sigma) = \sigma^{-1} \exp\{-e^{-(x-\mu)/\sigma} - (x-\mu)/\sigma\} \).

Let \( Y = (X - \mu)/\sigma \). Since \( \theta = (\mu, \sigma) \) is a location-scale parameter, the distribution of \( Y \) does not depend on \( \theta \). (It is \( G_3(y) \).) We have \( \partial Y/\partial \mu = -1/\sigma \) and \( \partial Y/\partial \sigma = -Y/\sigma \).

Using \( \ell = -\log \sigma - e^{-y} - y \), we find

\[
\frac{\partial \ell}{\partial \mu} = -\frac{e^{-y}}{\sigma} + \frac{1}{\sigma} \quad \text{and} \quad \frac{\partial \ell}{\partial \sigma} = -\frac{1}{\sigma} - \frac{ye^{-y}}{\sigma} + \frac{y}{\sigma}
\]

so that \( Ee^{-Y} = 1 \) and \( EY = EYe^{-Y} + 1 \). We also have

\[
\frac{\partial^2 \ell}{\partial \mu^2} = -\frac{e^{-y}}{\sigma^2}, \quad \frac{\partial^2 \ell}{\partial \mu \partial \sigma} = -\frac{ye^{-y}}{\sigma^2} + \frac{e^{-y}}{\sigma^2} - \frac{1}{\sigma^2} \quad \text{and} \quad \frac{\partial^2 \ell}{\partial \sigma^2} = \frac{1}{\sigma^2} - \frac{y^2 e^{-y}}{\sigma^2} + \frac{2ye^{-y}}{\sigma^2} - \frac{2y}{\sigma^2}.
\]

Taking expectations and simplifying, we find

\[
I(\mu, \sigma) = \frac{1}{\sigma^2} \begin{pmatrix} \frac{1}{EYe^{-Y}} & \frac{EYe^{-Y}}{EY^2 e^{-Y} + 1} \end{pmatrix} = \frac{1}{\sigma^2} \begin{pmatrix} 1 & -0.42278 \\ -0.42278 & 1.82367 \end{pmatrix}.
\]

If \( Z = -\log(Y) \), then \( Z \) has an exponential distribution with mean 1 and \( EYe^{-Y} = EY - 1 = \gamma - 1 \) and Maple tells us that \( EY^2 e^{-Y} = E(\log(Z)^2 Z) = (\pi^2/6) - 2\gamma + \gamma^2 \), from which the above numbers follow.

(b) If \( \hat{\theta}_n \) is an unbiased estimate of \( g(\mu, \sigma) = \mu/\sigma \), then with \( \dot{g}(\mu, \sigma) = (1/\sigma, -\mu/\sigma^2) \), we have

\[
\text{Var}_\theta(\hat{\theta}_n) \geq \frac{1}{n} \dot{g} \overline{I}^{-1} \dot{g}^T = \frac{1}{n} \left[ 1.1087 - 0.5140 \frac{\mu}{\sigma} + 0.6079 \frac{\mu^2}{\sigma^2} \right].
\]

(The minimum value occurs at \( \mu/\sigma = 0.4228 \).)