Solutions to Exercise Set 8.

14.4. (a) $\mathrm{P}(M_n \le x) = F(x)^n = \cos(x)^n$.

(b) For $x < 0$, $\mathrm{P}(M_n/b_n \le x) = \mathrm{P}(M_n \le b_n x) = F(b_n x)^n = \cos(b_n x)^n = (1 - (b_n x)^2/2 + \cdots)^n \to \exp\{\lim_{n\to\infty} -nb_n^2 x^2/2\}$. Thus, if we choose $b_n = \sqrt{2/n}$, we have

$$\sqrt{\frac{n}{2}} M_n \xrightarrow{\mathcal{L}} \exp\{-x^2\} \qquad \text{for} \quad x < 0.$$

This is a Weibull distribution on the negative axis, $(-\infty, 0)$.

17.4. (a)

$$p_n = P(H_1|X_1,\dots,X_n) = \frac{P(H_1,X_1,\dots,X_n)}{P(X_1,\dots,X_n)} = \frac{p_0\prod_1^n f_1(X_i)}{p_0\prod_1^n f_1(X_i) + (1-p_0)\prod_1^n f_0(X_i)}$$
$$= \left(1 + \frac{1-p_0}{p_0}\prod_1^n \frac{f_0(X_i)}{f_1(X_i)}\right)^{-1}.$$

(b) Assuming H_0 is true, we are to show

$$-\frac{1}{n}\log(p_n) = \frac{1}{n}\log\left(1 + \frac{1-p_0}{p_0}\prod_{i=1}^n \frac{f_0(X_i)}{f_1(X_i)}\right) \xrightarrow{a.s.} K(f_0,f_1).$$

Since $\frac{1}{n}\sum_1^n \log(f_0(X_i)/f_1(X_i)) \xrightarrow{a.s.} K(f_0,f_1) > 0$, we have $\prod_1^n (f_0(X_i)/f_1(X_i)) = \exp\{\sum_1^n \log(f_0(X_i)/f_1(X_i))\} = \exp\{n(\frac{1}{n}\sum_1^n \log(f_0(X_i)/f_1(X_i))\} \xrightarrow{a.s.} +\infty$. Therefore,

$$\frac{1}{n}\log\left(1 + \frac{1-p_0}{p_0}\prod_{i=1}^n \frac{f_0(X_i)}{f_1(X_i)}\right) \sim \frac{1}{n}\log\left(\frac{1-p_0}{p_0}\prod_{i=1}^n \frac{f_0(X_i)}{f_1(X_i)}\right)$$
$$= \frac{1}{n}\log\left(\frac{1-p_0}{p_0}\right) + \frac{1}{n}\sum_{i=1}^n \log\left(\frac{f_0(X_i)}{f_1(X_i)}\right) \xrightarrow{a.s.} K(f_0,f_1).$$

18.1. (a) The log likelihood function is proportional to $\theta^{nr}(1-\theta)^{\sum X_i}$. Hence, the MLE of θ is $\hat\theta_n = nr/(nr + \sum X_i) = r/(r + \overline{X}_n)$. To find Fisher information,

$$\frac{\partial}{\partial\theta}\log f(x|\theta) = \frac{r}{\theta} - \frac{x}{1-\theta} \qquad \text{and} \qquad \frac{\partial^2}{\partial\theta^2}\log f(x|\theta) = -\frac{r}{\theta^2} - \frac{x}{(1-\theta)^2}$$

From $\mathrm{E}_\theta \frac{\partial}{\partial\theta}\log f(X|\theta) = 0$ we find $\mathrm{E}_\theta X = r(1-\theta)/\theta$. So Fisher information is $\mathcal{I} = -\mathrm{E}_\theta \frac{\partial^2}{\partial\theta^2}\log f(x|\theta) = r/(\theta^2(1-\theta))$. Hence we have $\sqrt{n}(\hat\theta_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \theta^2(1-\theta)/r)$.

(b) Since the likelihood is proportional to $\theta^{\Sigma y_i}(1-\theta)^{nk-\Sigma y_i}$, the MLE for binomial sampling is $\tilde\theta_n = \overline{Y}_n/k$ and we have from the central limit theorem, $\sqrt{n}(\tilde\theta_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \theta(1-\theta)/k)$.

(c) Negative binomial sampling is better than binomial sampling if $\theta^2(1-\theta)/r < \theta(1-\theta)/k$, which reduces to $\theta < r/k$.