Solutions to Exercise Set 7.

13.1. (a) The distribution function and the density of the Cauchy distribution, $\mathcal{C}(\mu,\sigma)$ are

$$F(x) = \frac{1}{\pi}[\arctan((x-\mu)/\sigma) + \frac{\pi}{2}] \qquad f(x) = \frac{1}{\pi\sigma}\frac{1}{1+[(x-\mu)/\sigma]^2}.$$

The pth quantile satisfies $F(x_p) = p$, and solving for x_p we find $x_p = \mu + \sigma\tan(\pi(p-.5))$. Similarly, $x_{1-p} = \mu + \sigma\tan(\pi(.5-p))$. We note $f(x_p) = (\pi\sigma(1+[\tan(\pi(p-.5))]^2)^{-1} = f(x_{1-p})$, since $\tan(\pi(p-.5)) = -\tan(\pi(.5-p))$. Hence from the Corollary of Theorem 13, we have

$$\sqrt{n}\begin{pmatrix} X_{(np)} - x_p \\ X_{(n(1-p))} - x_{1-p}\end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(\begin{pmatrix}0\\0\end{pmatrix}, \pi^2\sigma^2(1+[\tan(\pi(.5-p))]^2)^2\begin{pmatrix} p(1-p) & p^2 \\ p^2 & p(1-p)\end{pmatrix})$$

(b) We apply Cramér's Theorem with $g(x_1,x_2) = (x_1+x_2)/2$ and $\dot{g}(x_1,x_2) = (.5,.5)$. We find

$$\sqrt{n}(\hat{\mu}_n - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \frac{\pi^2\sigma^2}{4}(1+[\tan(\pi(.5-p))]^2)^2 2p).$$

(c) We must find p to minimize $v(p) = (p/2)(1+[\tan(\pi(.5-p))]^2)^2$. Using Maple or Mathematica or some graphing calculator, we find that $v(p)$ is minimized for $0 < p \le .5$ by $p_{\min} = .440$, and the value there is $v(p_{\min}) = .2359$. This gives an asymptotic variance of $.2359\pi^2\sigma^2$ compared to the asymptotic variance of $.25\pi^2\sigma^2$ using the median, as in Example 13.2. Not much improvement.

13.2. (a) The pth quantile of F is $u_i + \theta$. So for $Z_i = X_{(\lceil np_i\rceil)} - u_i$, we have from the Corollary of Chapter 13, $\sqrt{n}(\boldsymbol{Z} - \theta\boldsymbol{1}) \xrightarrow{\mathcal{L}} \mathcal{N}(\boldsymbol{0}, \Sigma)$, where $\Sigma = (\sigma_{ij})$ and for $i \le j$, $\sigma_{ij} = p_i(1-p_j)/(f(u_i)f(u_j))$.

(b) Using Lagrange muliplier, write $\varphi(\boldsymbol{a}) = \boldsymbol{a}^T\Sigma\boldsymbol{a} - \lambda\boldsymbol{1}^T\boldsymbol{a}$. Then $\dot{\varphi}(\boldsymbol{a}) = 2\Sigma\boldsymbol{a} - \lambda\boldsymbol{1} = 0$ has solution $\boldsymbol{a} = \lambda\Sigma^{-1}\boldsymbol{1}/2$. To find λ, note that $1 = \boldsymbol{1}^T\boldsymbol{a} = \lambda\boldsymbol{1}^T\Sigma^{-1}\boldsymbol{1}/2$, so that $\lambda = 2/\,\boldsymbol{1}^T\Sigma^{-1}\boldsymbol{1}$. Therefore, $\boldsymbol{a} = \Sigma^{-1}\boldsymbol{1}\,/\,\boldsymbol{1}^T\Sigma^{-1}\boldsymbol{1}$.

(c) Let $g_i = f(u_i)$ for $i = 1,\ldots,k$, and let $p_0 = 0$ and $p_{k+1} = 1$. Then $\Sigma^{-1} = (\sigma^{ij})$, where

$$\sigma^{ij} = \begin{cases} \frac{(p_{i+1}-p_{i-1})g_i^2}{(p_{i+1}-p_i)(p_i-p_{i-1})} & \text{if } j = i \\ -\frac{g_ig_{i+1}}{(p_{i+1}-p_i)} & \text{if } j = i+1 \\ -\frac{g_ig_{i-1}}{(p_i-p_{i-1})} & \text{if } j = i-1 \\ 0 & \text{otherwise.}\end{cases}$$

(d) For the uniform distribution, $g_i = 1$ for all i, so the vector $\Sigma^{-1}\boldsymbol{1}$ is the transpose of $(1/p_1, 0, \ldots, 0, 1/(1-p_k))$, and $\hat{\theta} = (Z_1/p_1 - Z_k/(1-p_k))/(1/p_1 + 1/(1-p_k))$.