Solutions to Exercise Set 6.

12.1. We have $\bar{z}_N = (N+1)/2$, $\sigma_z^2 = (N^2-1)/12$ and $\max_j(z_j - \bar{z}_N)^2 = (N-1)^2/4$, as for the Rank-Sum test. We have that the Riemann sums, $\sum_{j=1}^N (1/\sqrt{j/N})(1/N)$ converge to the integral $\int_0^1 (1/\sqrt{x})\,dx = 2$, so $(1/\sqrt{N})\sum_1^N 1/\sqrt{j} \to 2$ or $\bar{a}_N \sim 2/\sqrt{N}$. Since $\sum_1^N 1/j \sim \log(N)$, we have $\sigma_a^2 \sim (\log(N)/N - (4/N)) \sim \log(N)/N$. We also have $\max_j(a(j) - \bar{a}_N)^2 \to 1$. Hence

$$\delta_n \sim N\frac{(N-1)^2/4}{N(N^2-1)/12}\,\frac{1}{\log(N)} \sim \frac{3}{\log(N)} \to 0$$

Thus condition (9) is satisfied and $(S_N - \mathrm{E}S_N)/\sqrt{\mathrm{Var}(S_N)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. Since $\mathrm{Var}(S_N) = N^2\sigma_z^2\sigma_a^2/(N-1) \sim N^2\log(N)/12$, we have

$$\frac{(S_N - N\bar{z}_N\bar{a}_N)}{\sqrt{N^2\log(N)/12}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \quad \text{or} \quad \frac{1}{\sqrt{\log(N)}}\left(\frac{S_N}{N} - \bar{z}_N\bar{a}_N\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1/12).$$

Note that $\bar{z}_N\bar{a}_N \sim \sqrt{N}$. One can also show that $|\sum_{j=1}^N (1/\sqrt{j/N})(1/N) - 2| \le 2/\sqrt{N}$. This allows us to show that $\bar{z}_N\bar{a}_N$ may be replaced by $\sqrt{N}$ to conclude

$$\frac{1}{\sqrt{\log(N)}}\left(\frac{S_N}{N} - \sqrt{N}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1/12) \quad \text{or} \quad \sqrt{\frac{N}{\log(N)}}\left(\frac{S_N}{N^{3/2}} - 1\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1/12)$$

12.5. (a) Since $\bar{a}_N = 0$, we have $\mathrm{E}S_N = 0$. The variance of S_N is $(N^2/(N-1))\sigma_z^2\sigma_a^2$, and since $\sigma_a^2 = (1/N)\sum_1^N a(i)^2 = 2n/N$, we have $\mathrm{Var}(S_N) = (2nN/(N-1))\sigma_z^2$.

(b) For asymptotic normality of S_N, we need

$$\frac{\max_j(z_j - \bar{z}_N)^2 \max(a(j) - \bar{a}_N)^2}{N\sigma_z^2\sigma_a^2} \to 0.$$

We have $\max_j(a(j) - \bar{a}_N)^2 = 1$, and $\sigma_a^2 = 2n/N$. Then the above condition becomes

$$\frac{\max_j(z_j - \bar{z}_N)^2}{2n\sigma_z^2} \to 0.$$