Solutions to Exercise Set 5.

10.3. (a) $\chi^2 = [(15-15)^2 + (21-15)^2 + (17-15)^2 + (7-15)^2]/15 = 104/15 = 6.93$. It has 3 degrees of freedom.

(b) $\lambda = [(12-15)^2 + (21-15)^2 + (18-15)^2 + (9-15)^2]/15 = 90/15 = 6.0$. At the 5% level of significance and 3 degrees of freedom, we see from the Fix tables that the power is only about .52.

(c) At 3 d.f., it requires a non-centrality parameter of 14.172 at 5% level to attain power .90. This requires $n(\lambda/60) = 14.172$, which reduces to $n = 141.72$; but since n must be an integer, $n = 142$.

11.3. Since $\sum |z_j| = \sum_{j=0}^{\infty} \beta^j = 1/(1-\beta) < \infty$ for $|\beta| < 1$, we have from Exercise 11.7 that $\sqrt{n}(\overline{Y}_n - \mu) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$. Here $\mu = \xi \sum_0^{\infty} \beta^j = \xi/(1-\beta)$, where $\xi = \mathrm{E}(X_i)$. Similarly with $\tau^2 = \mathrm{Var}(X_i)$, we find $\sigma_{00} = \tau^2/(1-\beta^2)$, and $\sigma_{0k} = \beta^k \tau^2/(1-\beta^2)$. Hence, $\sigma^2 = (\tau^2/(1-\beta^2))[1 + 2\beta + 2\beta^2 + 2\beta^3 + \cdots] = \tau^2/(1-\beta)^2$. So $\sqrt{n}(\overline{Y}_n - \xi) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tau^2/(1-\beta)^2)$.

12.2. (a) By symmetry, $z_j = -z_{N+1-j}$, so $\bar{z}_N = 0$. Then using the Riemann approximation to an integral, $\sigma_z^2 = (1/N)\sum z_j^2 = \sum \Phi^{-1}(j/(N+1))^2(1/N) \to \int_0^1 \Phi^{-1}(x)^2\,dx = \int_{-\infty}^{\infty} u^2\,d\Phi(u) = 1$, using the change of variable $u = \Phi^{-1}(x)$.

(b) We have $\max(a_j - \bar{a})^2 = \max\{(1-\bar{a})^2, \bar{a}^2)\} \to \max\{(1-r)^2, r^2\}$ and $\sigma_a^2 = (m/N)(1-(m/N)) \to r(1-r)$, and $\sigma_z^2 \to 1$. Therefore $\delta_N \to 0$ if and only if $\max z_j^2/N \to 0$. But $\max z_j^2 = z_N^2 = \Phi^{-1}(N/(N+1))^2$. We must show $\frac{1}{N}\Phi^{-1}(1 - \frac{1}{N+1})^2 \to 0$.

For large values of x, $1 - \Phi(x) \approx \phi(x)/x$, where $\phi(x) = \Phi'(x)$ is the normal density. This is seen using L'Hospital's rule. So for large x, $1 - \Phi(x) < \phi(x)$, or equivalently, $\Phi(x) > 1 - \phi(x))$ or equivalently, $x > \Phi^{-1}(1 - \phi(x))$. So for large N, $\Phi^{-1}(1 - \frac{1}{N+1}) < x$, where $(1/\sqrt{2\pi})e^{-x^2/2} = 1/(N+1)$. This shows for large N that $\Phi^{-1}(N/(N+1))^2 < x^2 < 2\log(N+1)$, and that $\frac{1}{N}\Phi^{-1}(1 - \frac{1}{N+1})^2 \to 0$.

Thus $\delta_N \to 0$ and Theorem 12 gives $S_N/\sqrt{\mathrm{Var}(S_N)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$, where $\mathrm{Var}(S_N) = N^2\sigma_z^2\sigma_a^2/(N-1) \approx Nr(1-r)$. Unlike Exercise 12.1(b) of the text, no further conditions are needed to conclude that $S_N/\sqrt{N} \xrightarrow{\mathcal{L}} \mathcal{N}(0, r(1-r))$.