Solutions to Exercise Set 4.

7.9 (a) We take $g(x) = \sqrt{x}$ and find $g'(x) = 1/(2\sqrt{x})$ and $g'(\sigma^2) = 1/(2\sigma)$. Hence, $\sqrt{n}(\sqrt{Z_n} - \sigma) \xrightarrow{\mathcal{L}} \mathcal{N}(0, (1/(2\sigma))^2 2\sigma^4) = \mathcal{N}(0, \sigma^2/2)$.

(b) We have $g''(x) = -1/(4x^{3/2})$, $g''(\sigma^2) = -1/(4\sigma^3)$ and

$$\gamma_n = \frac{\sqrt{n}g'(\sigma^2)}{\sqrt{2\sigma^4}g''(\sigma^2)} = \frac{\sqrt{n}(1/(2\sigma))}{\sqrt{2}\sigma^2(-1/4\sigma^3)} = -\sqrt{2n}.$$

Hence, $n(\sqrt{Z_n} - \sigma) \approx -(\sigma/4)[\chi_1^2(2n) - 2n]$.

(c) We find Z_n is $\mathcal{N}(\sigma^2, 2\sigma^4/n) = \mathcal{N}(1, 1/5)$, so $\mathrm{P}(Z_n < 0) = .0127$; we ignor this small probability that $\sqrt{Z_n}$ is imaginary. Then, $\mathrm{P}(\sqrt{10}(\sqrt{Z_n} - 1) > .5) = \mathrm{P}(\sqrt{Z_n} > 1.158) = \mathrm{P}(Z_n > 1.341) = \mathrm{P}(\sqrt{5}(Z_n - 1) > .762) = .223$ gives the exact probability. From (a), $\mathrm{P}(\sqrt{n}(\sqrt{Z_n} - 1) > .5) = \mathrm{P}(\sqrt{2n}(\sqrt{Z_n} - 1) > .707) = .240$. From (b), $10(\sqrt{Z_n} - 1) \approx -(1/4)[\chi_1^2(20) - 20]$, so $\mathrm{P}(\sqrt{10}(\sqrt{Z_n} - 1) > .5) \approx \mathrm{P}(-(1/4\sqrt{10})((Z + \sqrt{20})^2 - 20) > .5)$, where Z is $\mathcal{N}(0, 1)$. This reduces to $\mathrm{P}((Z + \sqrt{20})^2 < 20 - 2\sqrt{10}) = \mathrm{P}(-\sqrt{13.675} - \sqrt{20} < Z < \sqrt{13.675} - \sqrt{20}) = \mathrm{P}(-8.170 < Z < -.774) = .228$, much closer.

8.1.(a) Let us use the notation $\mu_{ij} = \mathrm{E}(X - \mu_x)^i(Y - \mu_y)^j$. Then from Theorem 8,

$$\sqrt{n}\left[\begin{pmatrix} s_x^2 \\ s_y^2 \end{pmatrix} - \begin{pmatrix} \sigma_x^2 \\ \sigma_y^2 \end{pmatrix}\right] \xrightarrow{\mathcal{L}} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu_{40} - \sigma_x^4 & \mu_{22} - \sigma_x^2\sigma_y^2 \\ \mu_{22} - \sigma_x^2\sigma_y^2 & \mu_{04} - \sigma_y^4 \end{pmatrix}\right)$$

Now use Cramér's Theorem with $g(x, y) = x/y$, and $\dot{g}(x, y) = (1/y, -x/y^2)$. Then $\sqrt{n}((s_x^2/s_y^2) - (\sigma_x^2/\sigma_y^2)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, v^2)$, where

$$\begin{aligned} v^2 &= \dot{g}(\sigma_x^2, \sigma_y^2)\begin{pmatrix} \mu_{40} - \sigma_x^4 & \mu_{22} - \sigma_x^2\sigma_y^2 \\ \mu_{22} - \sigma_x^2\sigma_y^2 & \mu_{04} - \sigma_y^4 \end{pmatrix}\dot{g}(\sigma_x^2, \sigma_y^2)^T \\ &= \frac{1}{\sigma_y^8}\left[\sigma_y^4\mu_{40} - 2\sigma_x^2\sigma_y^2\mu_{22} + \sigma_x^4\mu_{04}\right]. \end{aligned}$$

(b) If the parent distribution is bivariate normal with correlation coefficient ρ, then $\mu_{40} = 3\sigma_x^4$, $\mu_{22} = (1 + 2\rho^2)\sigma_x^2\sigma_y^2$, and $\mu_{04} = 3\sigma_y^4$. The limiting variance becomes $v^2 = 4(\sigma_x^4/\sigma_y^4)(1 - \rho^2)$.

(c) If $\rho \to \pm 1$, the variance goes to zero. If $\rho = \pm 1$, then $s_x^2/s_y^2 = \sigma_x^2/\sigma_y^2$ with probability 1. No normalization can make it converge to a nondegenerate limit.

9.3. (a) If $g(\boldsymbol{p}) = \boldsymbol{p}^T\boldsymbol{p}$, then $\dot{g}(\boldsymbol{p}) = 2\boldsymbol{p}^T$. Hence, $\sqrt{n}(g(\hat{\boldsymbol{p}}) - g(\boldsymbol{p})) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4\boldsymbol{p}^T\boldsymbol{\Sigma}\boldsymbol{p})$, where $\boldsymbol{\Sigma} = \boldsymbol{P} - \boldsymbol{p}\boldsymbol{p}^T$. The asymptotic variance is

$$4\boldsymbol{p}^T\boldsymbol{\Sigma}\boldsymbol{p} = 4(\boldsymbol{p}^T\boldsymbol{P}\boldsymbol{p} - \boldsymbol{p}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{p}) = 4(\sum_1^k p_i^3 - (\sum_1^k p_i^2)^2).$$

So, $\sqrt{n}(S(\hat{\boldsymbol{p}}) - S(\boldsymbol{p})) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4(\sum_1^k p_i^3 - S(\mathbf{p})^2))$.

(b) If $g(\boldsymbol{p}) = -\sum_1^k p_i \log p_i$, then $\dot{g}(\boldsymbol{p}) = -(1+\log p_1, \ldots, 1+\log p_k)$. Then $\sqrt{n}(H(\hat{\boldsymbol{p}}) - H(\boldsymbol{p})) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$, where the asymptotic variance is

$$\sigma^2 = \dot{g}(\boldsymbol{p})\Sigma\dot{g}(\boldsymbol{p})^T = \sum_1^k p_i(\log p_i)^2 - (\sum_1^k p_i \log p_i)^2.$$