Solutions to Exercise Set 4.

7.8. To find the asymptotic distribution of \( \hat{\sigma}^2 = m_2 - (m_1 m_3/m_2) \), we need the asymptotic joint distribution of \((m_1, m_2, m_3)\). From the central limit theorem with \(EX = \mu_1 = 0\), we have

\[
\sqrt{n} \left( \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu_2 \\ \mu_3 \end{pmatrix} \right) \xrightarrow{L} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathcal{F} \right)
\]

where

\[
\mathcal{F} = \begin{pmatrix}
\text{Var}(X) & \text{Cov}(X, X^2) & \text{Cov}(X, X^3) \\
\text{Cov}(X, X^2) & \text{Var}(X^2) & \text{Cov}(X^2, X^3) \\
\text{Cov}(X, X^3) & \text{Cov}(X^2, X^3) & \text{Var}(X^3)
\end{pmatrix}
\]

Now apply Cramér’s Theorem with \( g(m_1, m_2, m_3) = m_2 - (m_1 m_3/m_2) = \hat{\sigma}^2 \). We find \( \dot{g}(m_1, m_2, m_3) = (-m_2/m_3, 1 + (m_1 m_3/m_2^2), -m_1/m_2) \) and \( \dot{g}(0, \mu_2, \mu_3) = (-\mu_3/\mu_2, 1, 0) \). Using \( \text{Var}(X) = \mu_2 \), \( \text{Cov}(X, X^2) = \mu_3 \) and \( \text{Var}(X^2) = \mu_4 - \mu_2^2 \), we find \( \dot{g} \mathcal{F} \dot{g}^{-1} = \mu_4 - \mu_2^2 - (\mu_3^2/\mu_2) \). Therefore, \( \sqrt{n}(\hat{\sigma}^2 - \mu_2) \xrightarrow{L} \mathcal{N}(0, \tau^2) \), where \( \tau^2 = \mu_4 - \mu_2^2 - (\mu_3^2/\mu_2) \). This is less than or equal to \( \mu_4 - \mu_2^2 \) with equality if and only if \( \mu_3 = 0 \).

All two-point distributions with means zero are equivalent, up to change of scale, to one of the distributions, \( P(X = -1) = a/(a+1) \), \( P(X = a) = 1/(a+1) \), for some \( a > 0 \). We find

\[
EX^2 = \frac{a^2}{a+1} + \frac{a}{a+1} = a
\]

\[
EX^3 = \frac{a^3}{a+1} - \frac{a}{a+1} = a(a-1)
\]

\[
EX^4 = \frac{a^4}{a+1} + \frac{a}{a+1} = a^2(a-1)
\]

So \( \tau^2 = \mu_4 - \mu_2^2 - (\mu_3^2/\mu_2) = a(a^2 + a - 1) - a^2(a-1)^2 = 0 \).

8.2. (a) Let us denote the product \( a \times b \) central moment by \( \mu_{ab} = E[(X-\mu_x)^a(Y-\mu_y)^b] \), and its sample estimate by \( m_{ab} = (1/n) \sum_i (X_i - \bar{X}_n)^a(Y_i - \bar{Y}_n)^b \). From Theorem 8(a) we may deduce

\[
\sqrt{n} \left( \begin{pmatrix} s_x^2 \\ s_y^2 \end{pmatrix} - \begin{pmatrix} \sigma_x^2 \\ \sigma_y^2 \end{pmatrix} \right) \xrightarrow{L} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} C(XX, XX) & C(XX, YY) \\ C(XX, YY) & C(YY, YY) \end{pmatrix} \right)
\]

where \( C(XX, XX) = \mu_{40} - \sigma_x^4, C(YY, YY) = \mu_{04} - \sigma_y^4 \), and \( C(XX, YY) = \mu_{22} - \sigma_x^2 \sigma_y^2 \). Now we use Cramér’s Theorem with \( g(x, y) = \log(x/y) \) and \( \dot{g}(x, y) = (1/x, -1/y) \) to find

\[
\sqrt{n}(Z_n - \theta) \xrightarrow{L} \mathcal{N}(0, \frac{\mu_{40}}{\sigma_x^4} - 2 \frac{\mu_{22}}{\sigma_x^2 \sigma_y^2} + \frac{\mu_{04}}{\sigma_y^4}).
\]

(b) If we let \( \gamma^2 \) denote the variance of the limit of \( \sqrt{n}(Z_n - \theta) \), then we may estimate \( \gamma^2 \) by \( \hat{\gamma}^2 = b_{40} - 2b_{22} + b_{04} \), where \( b_{ab} = m_{ab}/(s_x^a s_y^b) \). Then an asymptotically distribution-free confidence interval for \( \theta \) at level \( 1 - \alpha \) is \( Z_n - z_{\alpha/2} \hat{\gamma}/\sqrt{n} < \theta < Z_n + z_{\alpha/2} \hat{\gamma}/\sqrt{n} \), where \( z_{\alpha/2} \) is the \( \alpha/2 \) cutoff point for the standard normal distribution.
9.6. X and Y are independent if and only if \( P(X = 1, Y = 1) = P(X = 1)P(Y = 1) \). This becomes \( p_{11} = (p_{11} + p_{12})(p_{11} + p_{21}) \), or \( p_{11} = p_{11}^2 + p_{11}p_{21} + p_{11}p_{12} + p_{12}p_{21} \), or \( p_{12}p_{21} = p_{11}(1 - p_{11} - p_{12} - p_{21}) = p_{11}p_{22} \). This is the equation \( \theta = 1 \).

(a) The asymptotic distribution of \( \hat{p} = (\hat{p}_{11}, \hat{p}_{12}, \hat{p}_{21}, \hat{p}_{22}) \) is given by

\[
\sqrt{n}(\hat{p} - p) \xrightarrow{\mathcal{L}} N(0, P - pp^T)
\]

so, using \( g(p) = p_{11}p_{22}/(p_{12}p_{21}) \), we have

\[
\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} N(0, \dot{g}(p)^T(P - pp^T)\dot{g}(p))
\]

One finds that \( \dot{g}(p)^T = \theta \left( \frac{1}{p_{11}}, \frac{-1}{p_{12}}, \frac{-1}{p_{21}}, \frac{1}{p_{22}} \right) \).

Therefore, \( \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} N(0, \sigma^2) \), where

\[
\sigma^2 = \dot{g}(p)^T(P - pp^T)\dot{g}(p) = \dot{g}(p)^TP\dot{g}(p) = \theta^2 \left( \frac{1}{p_{11}} + \frac{1}{p_{12}} + \frac{1}{p_{21}} + \frac{1}{p_{22}} \right)
\]

(b) Now let \( g(\theta) = \log(\theta) \), with \( \dot{g}(\theta) = 1/\theta \). We have

\[
\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} N(0, \sigma^2/\theta^2) = N(0, \left( \frac{1}{p_{11}} + \frac{1}{p_{12}} + \frac{1}{p_{21}} + \frac{1}{p_{22}} \right)).
\]