Stat 200C, Spring 2010 Ferguson

Solutions to Exercise Set 3.

5.5. (a) From the electronic distribution function calculators, we find $\mathrm{P}(X \le 10) = .58304$.

(b) $\mathrm{P}(X \le 10) = \mathrm{P}(X \le 10.5)$ (the correction for continuity). When $\lambda = 10$, $\mathrm{E}X = 10$ and $\mathrm{Var}X = 10$. So if $Z = (X - 10)/\sqrt{10}$, then $\mathrm{P}(X \le 10.5) = \mathrm{P}(Z \le z)$ where $z = (10.5 - 10)/\sqrt{10} = .15811$. From the normal distribution tables, we find $\mathrm{P}(Z \le z) \approx .56282$.

(c) For $X \in \mathcal{P}(\lambda)$, we find $\mathrm{E}X = \lambda$, $\mathrm{Var}X = \lambda$, $\mathrm{E}(X-\lambda)^3 = \lambda$, and $\mathrm{E}(X-\lambda)^4 = 3\lambda^2 + \lambda$. Hence $\beta_1 = 1/\sqrt{\lambda}$ and $\beta_2 = 1/\lambda$. The first Edgeworth correction term is $-\beta_1(z^2-1)\varphi(z)/6 = .02025$, so the corresponding estimate of the probability is $.56282 + .02025 = .58307$, amazingly close!

(d) The second Edgeworth correction term is $-\varphi(z)[3\beta_2(z^3 - 3z) + \beta_1^2(z^5 - 10z^3 + 15z)]/72 = -.00052$, so the corresponding estimate of the probability is $.58307 - .00052 = .58255$.

5.6. (a) We use the Cramér-Wold device of Exercise 3.2 of the text. Let $V_n = c_1 S_n + c_2 T_n = \sum_{j=1}^n (c_1 a_{nj} + c_2 b_{nj}) X_j$. We show V_n is asymptotically normal using the Lindeberg-Feller Theorem with $X_{nj} = (c_1 a_{nj} + c_2 b_{nj}) X_j$. We have $\mathrm{E}V_n = 0$ since $\mathrm{E}X_{nj} = 0$ and $B_n^2 = \mathrm{Var}(V_n) = \sum_{j=1}^n (c_1 a_{nj} + c_2 b_{nj})^2 = c_1^2 \sum a_{nj}^2 + 2c_1c_2 \sum a_{nj} b_{nj} + c_2^2 \sum b_{nj}^2 = c_1^2 + 2c_1c_2\rho_n + c_2^2 = (c_1, c_2)\begin{pmatrix} 1 & \rho_n \\ \rho_n & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. The Lindeberg condition is satisfied:

$$\begin{aligned}
\frac{1}{B_n^2}\sum_{j=1}^n \mathrm{E}(X_{nj}^2 \mathrm{I}(|X_{nj}| > B_n\epsilon)) &= \frac{1}{B_n^2}\sum_{j=1}^n (c_1 a_{nj} + c_2 b_{nj})^2 \mathrm{E}(X_j^2 \mathrm{I}(X_j^2 > \frac{B_n^2\epsilon^2}{(c_1 a_{nj} + c_2 b_{nj})^2})) \\
&\le \frac{1}{B_n^2}\sum_{j=1}^n (c_1 a_{nj} + c_2 b_{nj})^2 \mathrm{E}(X_j^2 \mathrm{I}(X_j^2 > \frac{B_n^2\epsilon^2}{\max_{i\le n}(c_1 a_{ni} + c_2 b_{ni})^2})) \\
&= \mathrm{E}(X_1^2 \mathrm{I}(X_1^2 > \frac{B_n^2\epsilon^2}{\max_{i\le n}(c_1 a_{ni} + c_2 b_{ni})^2})) \to 0,
\end{aligned}$$

since B_n^2 is bounded and $\max_{i\le n}(c_1 a_{ni} + c_2 b_{ni})^2 \to 0$. Thus $V_n/B_n \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. This implies $V_n = c_1 S_n + c_2 T_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, \mathbf{c}^T \mathbf{\Sigma} \mathbf{c})$, where $\mathbf{c} = (c_1, c_2)^T$ and $\mathbf{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. Then by the Cramér-Wold device, $(S_n, T_n) \xrightarrow{\mathcal{L}} \mathcal{N}((0,0), \mathbf{\Sigma})$.

(b) Let $S_n = \sum_{j=1}^n (1/\sqrt{n}) X_j$ and $T_n = \sum_{j=1}^n (j/B_n) X_j$, where $B_n^2 = \sum_1^n j^2 = n(n+1)(2n+1)/6$. Then $\rho_n = \sum_{j=1}^n (1/\sqrt{n})(j/B_n) \to \sqrt{3}/2 = \rho$. Also, $\max_j (1/\sqrt{n})^2 \to 0$ and $\max_{j\le n} j^2/B_n^2 \to 0$ so the conditions are satisfied. We conclude

$$(S_n, T_n) \simeq (\frac{1}{\sqrt{n}}\sum_1^n X_j, \frac{\sqrt{3}}{n^{3/2}}\sum_1^n jX_j) \xrightarrow{\mathcal{L}} \mathcal{N}((0,0), \begin{pmatrix} 1 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1 \end{pmatrix}).$$

6.3. Write $T_n = \sqrt{n}(X_n + Y_n Z_n - \mu_x - \mu_y \mu_z) = \sqrt{n}(X_n - \mu_x) + Z_n \sqrt{n}(Y_n - \mu_y) + \mu_y \sqrt{n}(Z_n - \mu_z)$. Then since $Z_n \xrightarrow{\mathrm{P}} \mu_z$, we have by Slutsky's Theorem

$$T_n \xrightarrow{\mathcal{L}} U + \mu_z V + \mu_y W \in \mathcal{N}(0, \sigma^2)$$

where $\sigma^2 = \sigma_x^2 + 2\mu_z \sigma_{xy} + \mu_z^2 \sigma_y^2 + 2\mu_y \sigma_{xz} + 2\mu_y \mu_z \sigma_{yz} + \mu_y^2 \sigma_z^2$.