Stat 200C, Spring 2009 Ferguson

**Solutions to Exercise Set 3.**

5.9 (a) Let $Z_n = \sum_{i=1}^n (X_{nj} - \mu_{nj})$, where $\mu_{nj} = \mathrm{E}X_{nj}$. Then the summands have mean zero, and $B_n^2 = \mathrm{Var}(Z_n) = \mathrm{Var}(S_n)$. Therefore

$$\frac{Z_n}{B_n} = \frac{S_n - \mathrm{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$$

provided the Lindeberg condition is satisfied. This is satisfied under the assumption that $B_n \to \infty$, because

$$\frac{1}{B_n^2}\sum_{j=1}^n \mathrm{E}[(X_{n,j} - \mu_{nj})^2 \mathrm{I}((X_{n,j} - \mu_{nj})^2 > \epsilon^2 B_n^2)] = 0$$

when $n$ is sufficiently large (when $B_n > 2A/\epsilon$, because $|X_{nj} - \mu_{nj}| \le |X_{nj}| + |\mu_{nj}| < 2A$ for all $n$ and $j$).

(b) Write $Y_n = \sum_1^n X_{nj}$ where $X_{nj}$ is Bernoulli with probability of success $p_n = 1/\sqrt{n}$. Then the $|X_{nj}|$ are uniformly bounded by 1, and $B_n = np_n = \sqrt{n} \to \infty$. The conditions are satisfied and $\dfrac{Y_n - np_n}{\sqrt{np_n(1-p_n)}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. This reduces to $\sqrt[4]{n}(\dfrac{Y_n}{\sqrt{n}} - 1) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$.

6.2 (a) Let $Y_i = |X_i|$. Then, $(X_i, Y_i)$ are i.i.d. with $\mathrm{E}(X_i, Y_i) = (0, \tau)$. Since $\mathrm{E}X^2 = 2\tau^2$, we have $\mathrm{Var}X_i = 2\tau^2$ and $\mathrm{Var}Y_i = \tau^2$. We also have $\mathrm{Cov}(X_i, Y_i) = 0$. Therefore, from the multivariate Central Limit Theorem,

$$\sqrt{n}(\overline{X}_n, (\overline{Y}_n - \tau)) \xrightarrow{\mathcal{L}} \mathcal{N}((0,0), \begin{pmatrix} 2\tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix}).$$

(b) From Theorem 6(a) with $f(x,y) = y/x$, continuous except on the line $x = 0$, we have $(\overline{Y}_n - \tau)/\overline{X}_n = \sqrt{n}(\overline{Y}_n - \tau)/(\sqrt{n}\overline{X}_n) \xrightarrow{\mathcal{L}} V/U$, where $U$ and $V$ are independent normal random variables with zero means and $2\tau^2$ and $\tau^2$ respectively. This has a Cauchy distribution with median zero and scale parameter $1/\sqrt{2}$ , independent of $\tau$.

7.4 (a) If $X$ has the given geometric distribution, the $\mathrm{E}X = \theta/(1-\theta)$ and $\mathrm{Var}X = \theta/(1-\theta)^2$. If $Y = \mathrm{I}(X > 0)$, then $\mathrm{E}Y = \theta$ and $\mathrm{Var}Y = \theta(1-\theta)$. Moreover, since $XY = X$, we have $\mathrm{E}XY = \mathrm{E}X$ so that $\mathrm{Cov}(X,Y) = \theta/(1-\theta) - \theta^2/(1-\theta) = \theta$. Then by the Central Limit Theorem,

$$\sqrt{n}\left(\begin{pmatrix} S_n/n \\ T_n/n \end{pmatrix} - \begin{pmatrix} \theta/(1-\theta) \\ \theta \end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\theta}{(1-\theta)^2} & \theta \\ \theta & \theta(1-\theta) \end{pmatrix}\right).$$

(b) We use Cramér's Theorem with $g(s,t) = (s/t, 1-t)$ applied to the vector, $(S_n/n, T_n/n)$. We have $g(\mathrm{E}X, \mathrm{E}Y) = (1/(1-\theta), 1-\theta)$. We find

$$\dot{g}(s,t) = \begin{pmatrix} \frac{1}{t} & -\frac{s}{t^2} \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \dot{g}(\mathrm{E}X, \mathrm{E}Y) = \begin{pmatrix} \frac{1}{\theta} & -\frac{1}{\theta(1-\theta)} \\ 0 & -1 \end{pmatrix}.$$

We conclude that

$$\sqrt{n}\left(\begin{pmatrix} S_n/T_n \\ 1 - T_n/n \end{pmatrix} - \begin{pmatrix} 1/(1-\theta) \\ 1-\theta \end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dot{g}\mathrm{Var}(X,Y)\dot{g}^T\right)$$
$$= \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{(1-\theta)^2} & 0 \\ 0 & \theta(1-\theta) \end{pmatrix}\right).$$