Solutions to Exercise Set 10.

22.2. (a) The likelihood function is $L = \prod_j \prod_i e^{-\lambda_j} \lambda_j^{x_{ij}}/x_{ij}! = \prod_j e^{-n\lambda_j} \lambda_j^{x_{\cdot j}}/c$ where $x_{\cdot j} = \sum_{i=1}^n x_{ij}$ and $c = \prod\prod x_{ij}!$. Under H_1, the MLE's are $\hat{\lambda}_j = x_{\cdot j}/n$. Under H_0, the likelihood is $L = \prod_j e^{-nj\lambda}(j\lambda)^{x_{\cdot j}}/c$, and the MLE is $\tilde{\lambda} = 2x_{\cdot\cdot}/(nk(k+1))$, where $x_{\cdot\cdot} = \sum_{j=1}^k x_{\cdot j}$. The likelihood ratio test rejects H_0 if the likelihood ratio,

$$\Lambda = \frac{\sup_{H_0} L}{\sup_{H_1} L} = \frac{e^{-x_{\cdot\cdot}} \prod(j\tilde{\lambda})^{x_{\cdot j}}/c}{e^{-x_{\cdot\cdot}} \prod \hat{\lambda}_j^{x_{\cdot j}}/c} = \frac{\prod(j\tilde{\lambda})^{x_{\cdot j}}}{\prod \hat{\lambda}_j^{x_{\cdot j}}}$$

is too small.

(b) Under H_0, the statistic $-2\log\Lambda$ has asymptotically a chi-square distribution with $k-1$ degrees of freedom. We reject H_0 if $-2\log\lambda_n \ge \chi^2_{k-1;\alpha}$.

24.2. (a) The modified chi-square is

$$\chi^2 = 100\Big[\frac{(.3-3\theta)^2}{.3} + \frac{(.5-(.5-\theta))^2}{.5} + \frac{(.1-(\frac{1}{3}-\theta))^2}{.1} + \frac{(.1-(\frac{1}{6}-\theta))^2}{.1}\Big]$$

Setting the derivative to zero gives a linear equation whose solution is $\tilde{\theta}_n = 3/26$.

(b) The modified χ^2 evaluated at this estimate becomes $\chi^2 = .710 + 2.663 + 13.910 + 2.374 = 19.657$. There are two degrees of freedom and the cutoff point at the 5% level of significance is $\chi^2_{2;.05} = 5.99$, so we reject the hypothesis. (The third cell is causing all the trouble.)

24.3. (a) Under H_0, the maximum likelihood estimates are $\tilde{p}_{11} = \tilde{p}_{22} = \tilde{p}_{33} = (n_{11} + n_{22} + n_{33})/3n$, and $\tilde{p}_{12} = \tilde{p}_{21} = \tilde{p}_{13} = \tilde{p}_{31} = \tilde{p}_{23} = \tilde{p}_{32} = (n_{12} + n_{21} + n_{13} + n_{31} + n_{23} + n_{32})/6n$, where n is the total sample size. The chi-square for testing H_0 is

$$\chi^2(\tilde{p}) = n \sum_{i=1}^3 \sum_{j=1}^3 \frac{((n_{ij}/n) - \tilde{p}_{ij})^2}{\tilde{p}_{ij}}.$$

(b) Originally there were 9 cells with 8 degrees of freedom, but we estimated 1 parameter and so lost one degree of freedom, ending with 7 degrees of freedom. We reject H_0 if $\chi^2(\tilde{p}) > \chi^2_{7;\alpha}$.

(c) The asymptotic distribution is noncentral $\chi^2_7(\lambda)$. The noncentrality parameter, λ, is found by replacing n_{ij}/n wherever it occurs in $\chi^2(\tilde{p})$ by the given p_{ij}. This means we also replace $\tilde{p}_{11} = \tilde{p}_{22} = \tilde{p}_{33}$ by .16 and $\tilde{p}_{12} = \tilde{p}_{21} = \tilde{p}_{13} = \tilde{p}_{31} = \tilde{p}_{23} = \tilde{p}_{32}$ by $[4(.10) + 2(.06)]/6 = .0867$. We find

$$\lambda = n\Big[3\frac{(.16-.16)^2}{.16} + 4\frac{(.10-.0867)^2}{.0867} + 2\frac{(.06-.0867)^2}{.0867}\Big] = .0246n.$$