1. Suppose that $X$ has a Poisson distribution with mean $\lambda$, $\mathcal{P}(\lambda)$, and that $X_n$ is a sequence of random variables such that $X_n \xrightarrow{L} X$. Is it necessarily true that
   (a) $X_n I(X_n > 0) \xrightarrow{L} X$?
   (b) $\sqrt{n}(X_n - \lambda) \xrightarrow{L} \mathcal{N}(0, \lambda)$?
   (c) $\operatorname{E} \left[ \frac{e^{X_n}}{1 + e^{X_n}} \right] \xrightarrow{L} \operatorname{E} \left[ \frac{e^{X}}{1 + e^{X}} \right]$?
   (For the “yes” answers, state what theorem are you using.)

2. Let $X_1, X_2, \ldots$ be independent random variables and that $X_j$ has a uniform distribution on the interval $(0, 2j)$. Let $S_n = \sum_{j=1}^{n} X_j$.
   (a) Find the mean and variance of $S_n$.
   (b) Show that $(S_n - \operatorname{E}(S_n))/\sqrt{\operatorname{Var}(S_n)} \xrightarrow{L} \mathcal{N}(0, 1)$, by checking the Lindeberg conditions.

3. Consider a multinomial distribution with 6 cells, sample size $n$, and probability vector $\mathbf{p} = \frac{1}{3}(p, p, p, 1 - p, 1 - p, 1 - p)$ for some unknown probability $p$, $0 < p < 1$. Let $n_j$ denote the number of observations that fall in cell $j$.
   (a) What is Pearson’s chi-square for testing the hypothesis $H_0 : p = .5$? What is its asymptotic distribution under this hypothesis?
   (b) What is Hellinger’s chi-square for testing $H_0$?
   (c) What is the approximate large sample distribution of Hellinger’s chi-square if the true value of $p$ is .4?

4. A waiting room contains a long line of $n$ chairs. As customers enter the room, they are seated in the chairs in order of arrival. Male customers appear with probability $p$ and female customers appear with probability $1 - p$, independently. Let $S_n$ denote the number of female customers that are seated between two male customers in the line.
   (a) Find the mean and variance of $S_n$.
   (b) Find is the asymptotic (for large $n$) distribution of $S_n$ properly normalized.

5. Let $X_1, X_2, \ldots, X_n$ be a sample from the distribution with distribution function $F(x) = 1 - (1/x)$ on the interval $(1, \infty)$.
   (a) Find the $p$th quantile, $x_p$ for this distribution, $0 < p < 1$.
   (b) Find the joint asymptotic distribution of the first and third sample terciles, $\hat{x}_{1/3}$ and $\hat{x}_{2/3}$, properly normalized.
1. (a) True. From Slutsky’s Theorem, since \( g(x) = xI(x > 0) \) is continuous.
(b) Very, very false.
(c) True. From the Helly-Bray Theorem, since \( g(x) = \exp(x)/(1 + \exp(x)) \) is bounded and continuous.

2. (a) Since \( X_j \) has the same distribution as \( 2jU_j \) where \( U_j \) has a uniform distribution on \((0,1)\), we have \( E(X_j) = j \) and \( \text{Var}(X_j) = \text{Var}(2jU_j) = 4j^2(1/12) = j^2/3 \). So \( E(S_n) = \sum_{j=1}^n j = n(n + 1)/2 \) and \( \text{Var}(S_n) = \sum_{j=1}^n j^2/3 = n(n + 1)(2n + 1)/18 \).
(b) Let \( X_{n,j} = X_j - j \), so that \( E(X_{n,j}) = 0 \). Then for \( B_n^2 = \text{Var}(S_n) \), we have \( Z_n/B_n = (S_n - E(S_n))/B_n \overset{L}{\to} \mathcal{N}(0,1) \) provided the Lindeberg condition holds. Using \( X_{n,j}^2 \leq n^2 \) for all \( j \leq n \), we have

\[
\frac{1}{B_n^2} \sum_{j=1}^\infty E[X_{n,j}^2 I(X_{n,j}^2 > e^2B_n^2)] \leq \frac{1}{B_n^2} \sum_{j=1}^\infty E[X_{n,j}^2 I(n^2 > e^2B_n^2)] = I(n^2 > e^2B_n^2) = 0 \quad \text{for all } n \text{ sufficiently large}
\]

3. (a) \( \chi_p^2 = n \sum_{j=1}^6 ((n_i/n) - (1/6))^2/(1/6) \). Under \( H_0 \), \( \chi_p^2 \) has asymptotically a chi-square distribution with 5 degrees of freedom.
(b) \( \chi_H^2 = 4n \sum_{i=1}^6 (\sqrt{n_i/n} - \sqrt{1/6})^2 \).
(c) If the true value of \( p \) is .4, then for large \( n \), \( \chi_H^2 \) has approximately a noncentral chi-square distribution with 5 degrees of freedom and noncentrality parameter \( \lambda \), where we may take

\[
\lambda = 4n \left[ 3 \left( \sqrt{\frac{2}{15}} - \sqrt{\frac{1}{6}} \right)^2 + 3 \left( \sqrt{\frac{1}{5}} - \sqrt{\frac{1}{6}} \right)^2 \right] = .04051n,
\]
or

\[
\lambda = 6n \left[ 3 \left( \frac{2}{15} - \frac{1}{6} \right)^2 + 3 \left( \frac{1}{5} - \frac{1}{6} \right)^2 \right] = \frac{n}{25}.
\]

4. (a) Let \( X_i = 1 \) if the \( i \)th customer is female and \( X_i = 0 \) otherwise. The \( X_i \) are i.i.d. Bernoulli with probability of success \( 1 - p \). Let \( Y_i = I(X_{i-1} = 0, X_i = 1, X_{i+1} = 0) \) for \( i = 2, \ldots, n - 1 \). The \( Y_i \) are 2-dependent and stationary, with \( EY_i = p^2(1 - p) \), \( \sigma_{00} = p^2(1 - p) - (p^2(1 - p))^2 \), \( \sigma_{01} = -(p^2(1 - p))^2 \), and \( \sigma_{02} = p^3(1 - p)^2 - (p^2(1 - p))^2 \). The
number of females seated between two males is \( S_n = \sum_{2}^{n-1} Y_i \). So \( \mathbb{E}S_n = (n-2)p^2(1-p) \) and \( \text{Var}(S_n) = (n-2)\sigma_{00} + 2(n-3)\sigma_{01} + 2(n-4)\sigma_{02} \).

(b) \( \sqrt{n}(S_n/(n-2) - p^2(1-p)) \xrightarrow{L} \mathcal{N}(0, \sigma^2) \), where

\[
\sigma^2 = \sigma_{00} + 2\sigma_{01} + 2\sigma_{02} = p^2(1-p) + 2p^3(1-p)^2 - 5(p^2(1-p))^2.
\]

5. (a) The \( p \)th quantile satisfies \( F(x_p) = p \), or \( 1 - (1/x_p) = p \). So \( x_p = 1/(1-p) \).

(b) \( x_{1/3} = 3/2 \) and \( x_{2/3} = 3 \). The density is \( f(x) = 1/x^2 \) for \( x > 1 \). So \( f(x_{1/3}) = f(3/2) = 4/9 \), and \( f(x_{2/3}) = f(3) = 1/9 \). Thus

\[
\sqrt{n} \left( \frac{\hat{x}_{1/3} - 3/2}{\hat{x}_{2/3} - 3} \right) \xrightarrow{L} \mathcal{N} \left( \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{cc} \frac{2/9}{(4/9)(1/9)} & \frac{1/9}{(4/9)(1/9)} \\ \frac{1/9}{(4/9)(1/9)} & \frac{2/9}{(1/9)^2} \end{array} \right) \right) \sim \mathcal{N} \left( \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{cc} 9/8 & 9/4 \\ 9/4 & 18 \end{array} \right) \right)
\]