Solution to Exercises 7.7.12 through 7.7.15.

7.7.12. (a) Let $Z_1, Z_2, \ldots$ be i.i.d. with $M(t) = \mathrm{E}e^{tZ}$, and suppose for some $0 < \rho < 1$ that $M(t_1) = M(t_2) = 1$ for some $t_1 \neq t_2$. Since M is convex and $M(0) = 1$, t_1 and t_2 are of opposite signs, so we take $t_1 < 0 < t_2$ without loss of generality. The Fundamental Identity gives

$$\mathrm{E}(e^{t_2 S_N}\rho^N) = \mathrm{E}(e^{t_1 S_N}\rho^N) = 1$$

For $N = \min\{n > 0 : a < S_n < b\}$, the Wald approximations are obtained by replacing S_N by a when $S_N \leq a$, and S_N by b when $S_N \geq b$.

$$e^{t_2 a}\mathrm{E}\rho^N \mathrm{I}(S_N \leq a) + e^{t_2 b}\mathrm{E}\rho^N \mathrm{I}(S_N \geq b) = 1$$
$$e^{t_1 a}\mathrm{E}\rho^N \mathrm{I}(S_N \leq a) + e^{t_1 b}\mathrm{E}\rho^N \mathrm{I}(S_N \geq b) = 1$$

Solving for $\mathrm{E}\rho^N \mathrm{I}(S_N \leq a)$ and $\mathrm{E}\rho^N \mathrm{I}(S_N \geq b)$ and summing gives

$$\mathrm{E}\rho^N = \mathrm{E}\rho^N \mathrm{I}(S_N \leq a) + \mathrm{E}\rho^N \mathrm{I}(S_N \geq b) = \frac{e^{t_1 a} - e^{t_2 a} + e^{t_2 b} - e^{t_1 b}}{e^{t_2 b + t_1 a} - e^{t_2 a + t_1 b}}.$$

The denominator is never zero since $t_2 b + t_1 a > 0$ and $t_2 a + t_1 b < 0$.

(b) For Example 1, where $Z_1 = z$ with probability p and $Z_1 = -z$ with probability $1 - p$ and we take $a = -jz$ and $b = kz$ for some j and k, there is no excess over the boundary, so the Wald approximation is exact. The equation for t_1 and t_2 is $M(t) = pe^{tz} + (1-p)e^{-tz} = 1/\rho$. This is quadratic in $x = e^{tz}$, with solutions

$$x_1(\rho) = e^{t_1 z} = \frac{1 - \sqrt{1 - 4\rho^2 p(1-p)}}{2\rho p}$$
$$x_2(\rho) = e^{t_2 z} = \frac{1 + \sqrt{1 - 4\rho^2 p(1-p)}}{2\rho p}.$$

Substituting this into the formula for $\mathrm{E}\rho^N$ of part (a), we find

$$\mathrm{E}\rho^N = \frac{x_1^{-j} - x_2^{-j} + x_2^k - x_1^k}{x_2^k x_1^{-j} - x_2^{-j} x_1^k}.$$

7.7.13. Using the fact that $\sum_1^\infty B_n(\theta)M_n(\theta)^{-n} = \mathrm{E}_\theta M_\theta(t)^{-N}\mathrm{I}(S_N \geq b)$ and $\sum_1^\infty A_n(\theta)M_n(\theta)^{-n} = \mathrm{E}_\theta M_\theta(t)^{-N}\mathrm{I}(S_N \leq a)$, we may write equation (7.89) as

$$1 \equiv \frac{(1-\theta)e^{tb}}{(1-\theta-t)}\mathrm{E}_\theta M_\theta(t)^{-N}\mathrm{I}(S_N \geq b) + \frac{(1+\theta)e^{ta}}{(1+\theta+t)}\mathrm{E}_\theta M_\theta(t)^{-N}\mathrm{I}(S_N \leq a)$$

We solve the equation $M_\theta(t) = 1/\rho$ for t, namely, $\rho(1-\theta^2) = 1 - (\theta+t)^2$, and find that there are two roots, $t_1 = -\theta - \sqrt{1 - \rho(1-\theta^2)}$ and $t_2 = -\theta + \sqrt{1 - \rho(1-\theta^2)}$ with $t_1 < 0 < t_2$. We have the two equations,

$$\frac{(1-\theta)e^{t_i b}}{(1-\theta-t_i)}\mathrm{E}_\theta\rho^N \mathrm{I}(S_N \geq b) + \frac{(1+\theta)e^{t_i a}}{(1+\theta+t_i)}\mathrm{E}_\theta\rho^N \mathrm{I}(S_N \leq a) = 1$$

for $i = 1, 2$. Solving for $\mathrm{E}_\theta\rho^N \mathrm{I}(S_N \geq b)$ and $\mathrm{E}_\theta\rho^N \mathrm{I}(S_N \leq a)$ and summing gives

$$\mathrm{E}_\theta\rho^N = \frac{\frac{(1+\theta)}{(1+\theta+t_1)}e^{t_1 a} - \frac{(1+\theta)}{(1+\theta+t_2)}e^{t_2 a} + \frac{(1-\theta)}{(1-\theta-t_2)}e^{t_2 b} - \frac{(1-\theta)}{(1-\theta-t_1)}e^{t_1 b}}{\frac{(1-\theta)}{(1-\theta-t_2)} \cdot \frac{(1+\theta)}{(1+\theta+t_1)}e^{t_2 b + t_1 a} - \frac{(1-\theta)}{(1-\theta-t_1)} \cdot \frac{(1+\theta)}{(1+\theta+t_2)}e^{t_1 b + t_2 a}}.$$

7.7.14. (a) For all $n \ge 1$, we have $\mathrm{P}(N \ge n) = \mathrm{P}(\epsilon_1 = 0, \ldots, \epsilon_{n-1} = 0, \hat{N} \ge n) = p_0^{n-1}\mathrm{P}(\hat{N} \ge n)$. Then assuming $p_0 < 1$,

$$\begin{aligned} \mathrm{E}N &= \sum_1^\infty n\mathrm{P}(N = 1) = \sum_1^\infty \mathrm{P}(N \ge n) = \sum_1^\infty p_0^{n-1}\mathrm{P}(\hat{N} \ge n) \\ &= \sum_1^\infty p_0^{n-1}\mathrm{P}(\hat{N} = n) + \sum_1^\infty p_0^{n-1}\mathrm{P}(\hat{N} \ge n+1) = \mathrm{E}p_0^{\hat{N}-1} + \sum_2^\infty p_0^{n-2}\mathrm{P}(\hat{N} \ge n) \\ &= \mathrm{E}p_0^{\hat{N}-1} + p_0^{-1}[\mathrm{E}N - 1]. \end{aligned}$$

Solving for $\mathrm{E}N$, we find $\mathrm{E}N = (1 - \mathrm{E}p_0^{\hat{N}})/(1 - p_0)$.

(b) The result follows immediately from the computations

$$\begin{aligned} \mathrm{P}(S_N = +\infty) &= \sum_1^\infty \mathrm{P}(N = n, S_n = +\infty) = \sum_1^\infty \mathrm{P}(\epsilon_1 = 0, \ldots, \epsilon_{n-1} = 0, \epsilon_n = \infty, \hat{N} \ge n) \\ &= \sum_1^\infty p_0^{n-1}p_+\mathrm{P}(\hat{N} \ge n) = p_+\mathrm{E}N = p_+(1 - \mathrm{E}p_0^{\hat{N}})/(1 - p_0) \end{aligned}$$

using (a). (Note the misprint in the text.)

(c) Similarly,

$$\begin{aligned} \mathrm{P}(b \le S_N < \infty) &= \sum_1^\infty \mathrm{P}(N = n, b \le S_n < \infty) = \sum_1^\infty \mathrm{P}(\epsilon_1 = 0, \ldots, \epsilon_{n-1} = 0, \epsilon_n = 0, \hat{N} = n, \hat{S}_n \ge b) \\ &= \sum_1^\infty p_0^n\mathrm{P}(\hat{N} = n, \hat{S}_n \ge b) = \sum_1^\infty \mathrm{P}(\hat{N} = n)p_0^n\mathrm{P}(\hat{S}_n \ge b|\hat{N} = n) = \mathrm{E}p_0^{\hat{N}}\mathrm{I}(\hat{S}_{\hat{N}} \ge b). \end{aligned}$$

7.7.15. We have

$$Z_i = \log\frac{f_1(X_i)}{f_0(X_i)} = \begin{cases} -\log 2 & \text{if } X_i = 0 \\ +\log 2 & \text{if } X_i = 1 \\ +\infty & \text{if } X_i = 2. \end{cases}$$

Under H_0, $\mathrm{P}(Z_i = -\log 2) = 4/5$ and $\mathrm{P}(Z_i = \log 2) = 1/5$, so we may use the formulas of Example 1, pp. 377-378, with $p = 1/5$ to find

$$\alpha_0 = \frac{1 - 4^{-j}}{4^k - 4^{-j}} \quad \text{and} \quad \mathrm{E}(N|H_0) = \frac{5}{3}\left[\frac{j(4^k - 1) - k(1 - 4^{-j})}{4^k - 4^{-j}}\right]$$

Under H_1, we may use Exercise 7.7.14 with $Z_i = \hat{Z}_i + \epsilon_i$, where

$$\hat{Z}_i = \begin{cases} -\log 2 & \text{with prob. } 1/2 \\ +\log 2 & \text{with prob. } 1/2 \end{cases} \quad \text{and} \quad \epsilon_i = \begin{cases} 0 & \text{with prob. } p_0 = 4/5 \\ +\infty & \text{with prob. } p_+ = 1/5 \end{cases}$$

and $p_- = 0$. To compute $\mathrm{E}\{N|H_1\}$ from Exercise 14(a), we need $\mathrm{E}p_0^{\hat{N}}$ which may be found from Exercise 12(b). Putting $\rho = p_0 = 4/5$ and $p = 1/2$ in the formulas for Exercise 12(b), we find $x_1(p_0) = 1/2$ and $x_2(p_0) = 2$ so that

$$\mathrm{E}p_0^{\hat{N}} = \frac{2^j - 2^{-j} + 2^k - 2^{-k}}{2^{j+k} - 2^{-(j+k)}} = \frac{(2^j + 2^k) - 2^{-(j+k)}(2^k + 2^j)}{(2^{j+k} + 1)(1 - 2^{-(j+k)})} = \frac{2^j + 2^k}{2^{j+k} + 1}$$

Then from Exercise 14(a),

$$\mathrm{E}\{N|H_1\} = \frac{1}{1 - p_0}(1 - \mathrm{E}p_0^{\hat{N}}) = 5\frac{(2^k - 1)(2^j - 1)}{2^{j+k} + 1}.$$

(Note the misprint in the text.) To compute $\alpha_1 = 1 - \mathrm{P}(S_N = +\infty) - \mathrm{P}(b \le S_N < \infty)$, first use 14(b) and the above to find

$$\mathrm{P}(S_N = +\infty) = p_+ \mathrm{E}(N|H_1) = \frac{1}{5} 5(1 - \mathrm{E}p_0^{\hat{N}}) = 1 - \mathrm{E}p_0^{\hat{N}}.$$

From 14(c), we must resolve the equations of 12(a) for $\mathrm{E}\rho^N \mathrm{I}(S_N \ge b)$ alone and solve the analog of 12(b). We find

$$\mathrm{P}(b \le S_N < \infty) = \mathrm{E}p_0^{\hat{N}} \mathrm{I}(\hat{S}_{\hat{N}} \ge b) = \frac{e^{t_1 a} - e^{t_2 a}}{e^{t_1 a + t_2 b} - e^{t_2 a + t_1 b}}$$

$$= \frac{x_1^{-j} - x_2^{-j}}{x_2^k x_1^{-j} - x_2^{-j} x_1^k} = \frac{2^j - 2^{-j}}{2^{j+k} - 2^{-(j+k)}}$$

since we have already found $x_1 = 1/2$ and $x_2 = 2$. Combining all this, we have

$$\alpha_1 = \mathrm{E}p_0^{\hat{N}} - \mathrm{P}(b \le S_N < \infty) = \frac{2^j - 2^{-j} + 2^k - 2^{-k}}{2^{j+k} - 2^{-(j+k)}} - \frac{2^j - 2^{-j}}{2^{j+k} - 2^{-(j+k)}} = \frac{2^k - 2^{-k}}{2^{j+k} - 2^{-(j+k)}}.$$