Solutions to Exercises 6.1.1 through 6.1.3.

6.1.1. $f(x|\theta) = \binom{5}{x}\theta^x(1-\theta)^{5-x}$ has monotone likelihood ratio, so, if the loss function satisfies (6.3) with $\theta_1 = 1/3$ and $\theta_2 = 2/3$, we want to find one-sided ϕ_j's such that $\mathrm{E}_{\theta_1}\phi_1(X) = \mathrm{E}_{\theta_1}\psi_1(X)$ and $\mathrm{E}_{\theta_2}\phi_2(X) = \mathrm{E}_{\theta_2}\psi_2(X)$, where $\psi_j(x) = \sum_{i=j+1}^{3}\psi(i|x)$ for $j = 1, 2$. The rule that improves on $\psi(j|x)$ is then $\phi(j|x) = \phi_{j-1}(x) - \phi_j(x)$. Note

$$\mathrm{E}_{\theta_2}\psi_2(X) = \mathrm{P}_{\theta_2}\{X = 2 \text{ or } X = 5\} = \binom{5}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3 + \left(\frac{2}{3}\right)^5 = \frac{72}{243}.$$

To get $\mathrm{E}_{\theta_2}\phi_2(X) = 72/243$, we start by putting $\phi_2(5) = 1$; this has probability $32/243$. We need $40/243$ more; if we put $\phi_2(4) = 1$, we would get $80/243$ more. Therefore, the desired rule is: $\phi(3|x) = \phi_2(x) = 0$ for $x \le 4$, $\phi(3|4) = 1/2$, and $\phi(3|5) = 1$. By symmetry, $\phi(1|x) = 0$ for $x \ge 2$, $\phi(1|1) = 1/2$, and $\phi(1|0) = 1$. We find $\phi(2|x)$ by subtracting the sum from 1: $\phi(2|0) = \phi(2|5) = 0$, $\phi(2|1) = \phi(2|4) = 1/2$, and $\phi(2|3) = \phi(2|4) = 1$.

6.1.2. (a) Consider the rule ϕ_a defined for $a \ge 0$ as

$$\begin{aligned}\phi_a(1|x) &= \mathrm{I}_{(-\infty,-a)}(x),\\ \phi_a(2|x) &= \mathrm{I}_{(-a,a)}(x),\\ \phi_a(1|x) &= \mathrm{I}_{(a,\infty)}(x).\end{aligned}$$

This rule has risk function

$$\begin{aligned}R(\theta,\phi_a) &= (\theta+1)^2\mathrm{P}_\theta(X < -a) + \theta^2\mathrm{P}_\theta(-a \le X \le a) + (\theta-1)^2\mathrm{P}_\theta(a < X)\\ &= \theta^2 + (2\theta+1)\Phi(-\theta-a) + (1-2\theta)(1-\Phi(-\theta+a))\end{aligned}$$

where $\Phi(z)$ is the distribution function of $\mathcal{N}(0,1)$. We now show that for each θ in $[-1,1]$ $R(\theta,\phi_a)$ is a decreasing function of a for $a < .549\cdots$. This shows that ϕ_a is not admissible for any $a < .549\cdots$. The derivative of R with respect to a,

$$\frac{\partial}{\partial a}R(\theta,\phi_a) = -(2\theta+1)\frac{1}{\sqrt{2\pi}}e^{-(\theta+a)^2/2} - (2\theta-1)\frac{1}{\sqrt{2\pi}}e^{-(\theta-a)^2/2}$$

is negative if and only if

$$2\theta[e^{a\theta} - e^{-a\theta}] < [e^{a\theta} + e^{-a\theta}]$$

which holds if and only if $2\theta\tanh(a\theta) < 1$. This is symmetric in θ so we restrict θ to be in the interval $[0,1]$. Since $\tanh(z)$ is an increasing function, we may solve for a to find $a < \tanh^{-1}(1/2\theta)/\theta$, where we think of $\tanh^{-1}(1/2\theta)$ as $+\infty$ if $\theta \le 1/2$. But $\tanh^{-1}(1/2\theta)/\theta$ is otherwise decreasing in θ so the worst case occurs at $\theta = 1$. Hence, $R(\theta,\phi_a)$ is a decreasing function of a for each θ in $[-1,1]$ provided $a < \tanh^{-1}(1/2) = .549\cdots$.

(b) Now suppose that $\Theta = (-\infty,\infty)$. To show that every monotone rule is admissible, it is sufficient to show that for a given monotone ϕ there does not exist a better monotone ψ, because Theorem 6.1.1 implies that the monotone rules form an essentially complete class for this problem. Suppose that

$$\begin{aligned}&\psi(1|x) = \mathrm{I}_{(-\infty,a_1)}(x) \qquad \psi(2|x) = \mathrm{I}_{(a_1,b_1)}(x) \qquad \psi(3|x) = \mathrm{I}_{(b_1,\infty)}(x)\\ &\phi(1|x) = \mathrm{I}_{(-\infty,a_2)}(x) \qquad \phi(2|x) = \mathrm{I}_{(a_2,b_2)}(x) \qquad \phi(3|x) = \mathrm{I}_{(b_2,\infty)}(x)\end{aligned}$$

and that ψ is as good as ϕ, where $a_1 < b_1$ and $a_2 < b_2$. In terms of the risk functions, we have for all θ,

$$\begin{aligned}&(\theta+1)^2\mathrm{P}_\theta(X < a_1) + \theta^2\mathrm{P}_\theta(a_1 < X < b_1) + (1-\theta)^2\mathrm{P}_\theta(b_1 < X)\\ &\qquad\le (\theta+1)^2\mathrm{P}_\theta(X < a_2) + \theta^2\mathrm{P}_\theta(a_2 < X < b_2) + (1-\theta)^2\mathrm{P}_\theta(b_2 < X)\end{aligned}$$

or, subtracting θ^2 from both sides,

$$(2\theta+1)\mathrm{P}_\theta(X<a_1)+(1-2\theta)\mathrm{P}_\theta(X>b_1)\le(2\theta+1)\mathrm{P}_\theta(X<a_2)+(1-2\theta)\mathrm{P}_\theta(X>b_2). \qquad (*)$$

We use the fact that the normal tails decrease to zero very rapidly, namely, $\mathrm{P}_0(X>x)\sim(1/x)e^{-x^2/2}/\sqrt{2\pi}$. This follows from

$$xe^{x^2/2}\int_x^\infty e^{-z^2/2}\,dz=x\int_0^\infty e^{-u}(x^2+2u)^{-1/2}\,du=\int_0^\infty e^{-u}(1+2(u/x^2))^{-1/2}\,du\to 1$$

as $x\to\infty$. Suppose that $a_1<a_2$. Then subtract $2\theta+1$ from both sides of (*), divide both sides by $\mathrm{P}_\theta(X>a_1)$ and let $\theta\to-\infty$. The left side tends to $+\infty$ and the right side tends to zero. This contradiction shows that $a_1\ge a_2$. Similarly, letting $\theta\to+\infty$ shows that $b_1\le b_2$. Now suppose that $a_2<a_1$. Then (*) implies that $b_1\ne b_2$, and (*) reduces to

$$(2\theta+1)\mathrm{P}_\theta(a_2<X<a_1)+(1-2\theta)\mathrm{P}_\theta(b_1<X<b_2)\le 0.$$

This inequality is clearly false for any θ such that $\mathrm{P}_\theta(a_2<X<a_1)=\mathrm{P}_\theta(b_1<X<b_2)$. But such a θ exists since both sides of this equation are continuous in θ and the left side is less than (resp. greater than) the right for θ sufficiently close to $-\infty$ (resp. $+\infty$). Hence $a_1=a_2$ and by symmetry, $b_1=b_2$.

6.1.3. (a) We must find the value of x_2 that minimizes (6.10). The left side of the max in (6.10) is decreasing in x_2 for all θ_0 and, if $L=1$, the right side is increasing in x_2. This max is minimized therefore when x_2 is chosen so that the left side is equal to the right side, that is when $\mathrm{P}_{\theta_0}\{X<x_2\}+\mathrm{P}_{\theta_0}\{X>x_2\}=\mathrm{P}_{\theta_0}\{X<x_2\}$. For the logistic distribution, this is

$$(1+e^{-(-x_2-\theta_0)})^{-1}+1-(1+e^{-(x_2-\theta_0)})^{-1}=(1+e^{-(x_2-\theta_0)})^{-1}.$$

This reduces to $2e^{-x_2}+e^{\theta_0}=e^{x_2}$. For given θ_0, this may be solved numerically for x_2. For example, if $\theta_0=.68437\ldots$, then $x_2=1.0000\ldots$, and if $\theta_0=1.96268\ldots$, then $x_2=2.0000\ldots$.

(b) For $L=2$, the minimum of (6.10) occurs when $x_2=\theta_0$.

(c) We will show the risk is made smaller if we increase x_2 slightly. The risk function of the rule (6.9) with $x_1=-x_2$ as given above (6.10) becomes for the logistic distribution

$$R(\theta,\phi)=\begin{cases}(1+e^{x_2+\theta})^{-1}+1-(1+e^{-x_2+\theta})^{-1} & \text{if } 0\le\theta\le\theta_0\\ (L-1)(1+e^{x_2+\theta})^{-1}+(1+e^{-x_2+\theta})^{-1} & \text{if } \theta>\theta_0.\end{cases}$$

For all $\theta\le\theta_0$, the risk is decreasing in x_2. We complete the proof by showing that for all $\theta>\theta_2$, the derivative of R with respect to x_2 is negative if L is sufficiently large. For $\theta>\theta_2$, this derivative is

$$\frac{\partial R}{\partial x_2}=-\frac{(L-1)e^{x_2+\theta}}{(1+e^{x_2+\theta})^2}+\frac{e^{-x_2+\theta}}{(1+e^{-x_2+\theta})^2}.$$

This is nonpositive if $(L-1)e^{x_2+\theta}(1+e^{-x_2+\theta})^2\ge e^{-x_2+\theta}(1+e^{x_2+\theta})^2$, which holds if

$$(L-1)(e^{2x}+2e^{x_2+\theta}+e^{2\theta})\ge 1+2e^{x_2+\theta}+e^{2x_2+2\theta}.$$

If $L>2$, the first two terms on the left dominate the first two terms on the right, and if $L-1\ge e^{2x_2}$, the third term on the left also dominates the third term on the right for all θ, completing the proof. Note, however, that the inequality in the text, $L-1\ge e^{x_2}$, must be corrected to $L-1\ge e^{2x_2}$.