Solutions to the Exercises of Section 5.7.

5.7.1. If $X_1,\ldots,X_N$ are i.i.d. from the density $f(x|\theta)=\theta e^{\theta x}I_{(-\infty,0)}(x)$, and if $V_{(1)}<V_{(2)}<\cdots<V_{(N)}$ denote the order statistics, then the joint density of $V_{(1)},\cdots,V_{(N)}$ is

$$f_{V_{(1)},\cdots,V_{(N)}}(v_1,\ldots,v_N|\theta)=N!\theta^N\exp\{\theta\sum_{j=1}^N v_j\}$$

for $-\infty<v_1<\cdots<v_N<0$. Let $Y_j=V_{(j)}-V_{(j+1)}$ for $j=1,\ldots,N-1$ and let $Y_N=V_{(N)}$. The inverse transformation is $V_{(j)}=Y_j+\cdots+Y_N$ for $j=1,\ldots,N$. The Jacobian of this transformation is $+1$, so the joint density of $Y_1,\ldots,Y_n$ is

$$f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_N|\theta)=N!\theta^N\exp\{\theta\sum_{j=1}^N jy_j\}$$

for $-\infty<y_j<0$ for $j=1,\ldots,N$. Since this density factors into

$$f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_N|\theta)=\prod_{j=1}^N j\theta\exp\{j\theta y_j\}I_{(-\infty,0)}(y_j),$$

we see that $Y_1,\ldots,Y_N$ are independent and the density of Y_j is $f_{Y_j}(y_j|\theta)=j\theta\exp\{j\theta y_j\}I_{(-\infty,0)}(y_j)$.

5.7.2. We must compute $\mathrm{E}\exp\{(\theta_1-\theta_0)\sum_1^m V_{(r_i)}\}$, where $V_{(1)},\ldots,V_{(N)}$ are the order statistics of a sample of size $N=m+n$ from the distribution with density $f(x|\theta_0)$. The inverse transformation to the transformation $Y_1=V_{(1)}-V(2),\ldots,Y_{N-1}=V_{(N-1)}-V_{(N)},Y_N=V_{(N)}$, is found to be $V_{(i)}=\sum_{j=i}^N Y_j$ for $i=1,\ldots,N$. From this we find

$$\sum_{i=1}^m V_{(r_i)}=\sum_{i=1}^m\sum_{j=r_i}^N Y_j=\sum_{j=1}^N c_jY_j,$$

where for $j=1,\ldots,N$, $c_j=k$ if $r_k\le j<r_{k+1}$. Since the Y_j are independent, we may write $\mathrm{E}\exp\{(\theta_1-\theta_0)\sum_1^m V_{(r_i)}\}=\mathrm{E}\exp\{(\theta_1-\theta_0)\sum_1^N c_jY_j\}=\prod_{j=1}^N\mathrm{E}\exp\{(\theta_1-\theta_0)c_jY_j\}$.

In the special case $\theta_0=1$ and $\theta_1=2$, we may use $\mathrm{E}\exp\{c_jY_j\}=j/(j+c_j)$ to find

$$\mathrm{E}\exp\{(\theta_1-\theta_0)\sum_{i=1}^m V_{(r_i)}\}=\prod_{j=1}^N\frac{j}{j+c_j}=\frac{N!r_1(r_2+1)\cdots(r_m+M-1)}{(N+m)!}$$

From (5.107), a most powerful rank test (of $H_0: f(x)=g(x)=f(x|\theta_0)$ against $H_1: f(x)=f(x|\theta_1)$ and $g(x)=f(x|\theta_0)$) is to reject H_0 if $r_1(r_2+1)\cdots(r_m+m-1)$ is too large. The distribution function of $f(x|\theta_0)$ is $G(x)=e^x$ for $x<0$ and the distribution function of $f(x|\theta_1)$ is $F(x)=e^{2x}$ for $x<0$. Then, since $F(x)=G(x)^2$, the above test is also a most powerful rank test of $H_0: F=G$ against the alternatives $H_1: F=G^2$.

5.7.3. The density of $V_{(r)}$ at v is the probability that one of the N observations falls at v, namely $N\,dv$, times the probability that of the remaining $N-1$ observations exactly $r-1$ fall to the left of v and the other $N-r$ fall to the right of v. This gives

$$f_{V_{(r)}}(v)=N\binom{N-1}{r-1}v^{r-1}(1-v)^{N-r},$$

which is the density of the beta, $\mathcal{B}e(r,N-r+1)$. Hence, $\mathrm{E}V_{(r)}=r/(N+1)$.

5.7.4. For the logistic distribution with $g(x) = e^x/(1+e^x)^2$, the distribution function is $G(x) = e^x/(1+e^x)$. We have $g'(x)/g(x) = (d/dx)\log g(x) = (d/dx)(x - 2\log(1+e^x)) = 1 - 2e^x/(1+e^x) = 1 - 2G(x)$. Hence, the test (5.110) becomes

$$\phi(\mathbf{r}) = \begin{cases} 1 & \text{if} \quad \sum_{i=1}^m \mathrm{E}G(V_{(r_i)}) & > \text{some } K, \\ \gamma & & = \\ 0 & & < \end{cases}$$

But given a sample $V_1, \ldots, V_N$ from a distribution $G(v)$, the variables $G(V_1), \ldots, G(V_N)$ form a sample from the uniform distribution, $\mathcal{U}(0,1)$, and the order statistics are $G(V_{(1)}), \ldots, G(V_{(N)})$. Hence, from Exercise 5.7.3, $\mathrm{E}G(V_{(r)}) = r/(N+1)$, and the above test rejects H_0 if $\sum_{i=1}^m R_i$ is too large. This is the Wilcoxon test.

5.7.5. Since this is a location parameter family we may use (5.110). We find $-g'(x)/g(x) = 1$ if $x > 0$ and $-g'(x)/g(x) = -1$ if $x < 0$. Therefore the locally best test (5.110) rejects H_0 if $\sum_{i=1}^m [2\mathrm{P}(V_{(r_i)} > 0) - 1]$ is too large, or equivalently, if $\sum_{i=1}^m \mathrm{P}(V_{(r_i)} > 0)$ is too large. But $\mathrm{P}(V_{(r)} > 0)$ is just the probability that at most $r-1$ of the N observations fall below zero, and since 0 is the median of the distribution, this is just the probability of at most $r-1$ successes in N independent trials with probability $1/2$ of success on each trial. Using the normal approximation for large N, we have that $\mathrm{P}(V_{(r)} > 0)$ is approximately $\Phi((r - (N+1)/2)/\sqrt{N/4})$, using a correction for continuity. This leads to the test that rejects H_0 if $\sum_{i=1}^m \Phi((R_i - (N+1)/2)/\sqrt{N/4})$ is too large.

5.7.6. Let $f(x|\theta) = c(\theta)h(x)\mathrm{I}_{(\theta,\infty)}(x)$. We are given a sample $Y_1, \ldots, Y_n$ from $g(x) = f(x|\theta_0)$ and a sample $X_1, \ldots, X_m$ from $f(x) = f(x|\theta)$. We are to test the hypothesis $H_0 : \theta = \theta_0$ against the hypothesis $H_1 : \theta > \theta_0$ based on the ranks $R_1, \ldots, R_m$ of the X's. Note that $g(x) = 0$ implies that $f(x) = 0$, so that Theorem 1 gives the joint distribution of the R_i:

$$\begin{aligned} \mathrm{P}(R_1 = r_1, \ldots, R_m = r_m|\theta) &= \binom{m+n}{m}^{-1} \left(\frac{c(\theta)}{c(\theta_0)}\right)^m \mathrm{E}\prod_{i=1}^m \frac{I_{(\theta,\infty)}(V(r_i))}{I_{(\theta_0,\infty)}(V(r_i))} \\ &= \binom{m+n}{m}^{-1} \left(\frac{c(\theta)}{c(\theta_0)}\right)^m \mathrm{P}(V_{(r_1)} > \theta) \end{aligned}$$

where the $V(r)$ are the order statistics of the combined sample under H_0. Under H_0, $\mathrm{P}(R_1 = r_1, \ldots, R_m = r_m|\theta) = \binom{m+n}{m}^{-1}$. By the Neyman-Pearson Lemma, the best test of H_0 against any simple hypothesis in H_1 has the form

$$\phi(\mathbf{r}) = \begin{cases} 1 & \text{if } \mathrm{P}(V_{(r_1)} > \theta) > K \\ \gamma & \text{if } \mathrm{P}(V_{(r_1)} > \theta) = K \\ 0 & \text{if } \mathrm{P}(V_{(r_1)} > \theta) < K. \end{cases}$$

Since $\mathrm{P}(V_{(r_1)} > \theta)$ is nondecreasing in r_1, this is equivalent to

$$\phi(\mathbf{r}) = \begin{cases} 1 & \text{if } r_1 > K' \\ \gamma & \text{if } r_1 = K' \\ 0 & \text{if } r_1 < K'. \end{cases}$$

5.7.7. First, suppose $F(x|\theta) = (e^{\theta G(x)} - 1)/(e^\theta - 1)$. Since the power of the test depends only on $\psi(z) = (e^{\theta z} - 1)/(e^\theta - 1)$, we may take $G(x)$ as we please. If we take $G(x)$ to be the uniform distribution on $(0,1)$, then $f(x|\theta)$ is of the form (5.106) and so the locally optimal rank test is of the form (5.109). In fact, this is just the Wilcoxon test as described below (5.109).

Suppose now that $F(x|\theta) = G(x)/(e^\theta(1 - G(x)) + G(x))$. Again we may choose $G(x)$ as we like. If we choose it to be the logistic distribution, $G(x) = e^x/(1+e^x)$, then $F(x|\theta) = e^{x-\theta}/(1+e^{x-\theta})$ is just the logistic with location parameter θ. So this is just the problem solved in Exercise 5.7.4, and the locally best rank test is again the Wilcoxon test.