Solutions to the Exercises of Section 4.7.

4.7.1. If the coin comes up heads, then d_2 is minimax. It guarantees the statistician a loss of (at most) $1-\epsilon$, and nature by choosing $\theta = 1$ can guarantee the statistician's loss to be at least $1-\epsilon$. Similarly, if the coin comes up tails, then d_4 is minimax. Nature can guarantee the statisticain's loss is at least $1-\epsilon$ by choosing $\theta = 0$. The combined rule, choose d_2 if the coin comes up heads, and d_4 if the coin comes up tails, guarantees a loss of (at most) $1-\epsilon$, but is not minimax or admissible in the unconditional problem. The opposite rule, choose d_1 if the coin comes up heads, and d_3 if the coin comes up tails, guarantees an expected loss of (at most) $1/2 < 1-\epsilon$, no matter what the true value of θ is. In the unconditional problem, it is assumed that nature cannot choose θ based on the outcome of the toss.

4.7.2. All three problems are invariant under location change. The best invariant estimate of θ is $d_0(\mathbf{X}) = X_1 - b_0(\mathbf{Y})$, where $Y_2 = X_2 - X_1, \ldots, Y_n = X_n - X_1$, and $b_0(\mathbf{Y})$ is that number that minimizes $\mathrm{E}_0(L(X_1 - b_0)|\mathbf{Y})$. We find the conditional density of X_1 given $\mathbf{Y}$ when $\theta = 0$ using (4.56). The joint density of the X_i when $\theta = 0$ is

$$f(x_1, x_2, \ldots, x_n) = \prod_{1\le i\le n} \mathrm{I}(-\frac{1}{2} < x_i < \frac{1}{2}) = \mathrm{I}(-\frac{1}{2} < \min_{1\le i\le n} x_i < \max_{1\le i\le n} x_i < \frac{1}{2}).$$

The joint density of $X_1, Y_2, \ldots, Y_n$ when $\theta = 0$ is

$$f(x_1, y_2 + x_1, \ldots, y_n + x_1) = \mathrm{I}(-\frac{1}{2} < x_1 + \min_{1\le i\le n} y_i < x_1 + \max_{1\le i\le n} y_i < \frac{1}{2})$$

where we are using the dummy variable $y_1 = 0$. The conditional density of X_1 given $Y_2, \ldots, Y_n$ is this divided by a function of $y_2, \ldots, y_n$ (the marginal density). So we see that this conditional density is the uniform density on the interval $(-\frac{1}{2} - \min_{1\le i\le n} y_i, \frac{1}{2} - \max_{1\le i\le n} y_i)$,

$$f_{X_1|Y_2=y_2,\ldots,Y_n=y_n}(x_1|0) = \frac{\mathrm{I}(-\frac{1}{2} - \min_{1\le i\le n} y_i < x_1 < \frac{1}{2} - \max_{1\le i\le n} y_i)}{1 - \min_{1\le i\le n} y_i - \max_{1\le i\le n} y_i}.$$

(a) For squared error loss, the best invariant estimate of θ is X_1 minus the mean of this distribution. This mean is the midpoint of the interval $(-\frac{1}{2} - \min_{1\le i\le n} y_i, \frac{1}{2} - \max_{1\le i\le n} y_i)$, so that

$$d_0(\mathbf{X}) = X_1 + \frac{1}{2}(\min_{1\le i\le n} Y_i + \max_{1\le i\le n} Y_i) = \frac{1}{2}(\min_{1\le i\le n} X_i + \max_{1\le i\le n} X_i).$$

(b) For absolute error loss, the best invariant estimate is X_1 minus the median of this distribution. This gives the same estimate as in (a).

(c) For $L(a,\theta) = \mathrm{I}(|a-\theta| > c)$, the best invariant estimate is X_1 minus the midpoint of the modal interval of length $2c$ (see Exercise 1.8.5). Since the density is flat, there may be many such modal intervals. But since the density is centered at the mean, we may always use the mean as the midpoint of the modal interval of length $2c$ for any value of $c > 0$. This leads to the esitmate of part (a) as a best invariant estimate. However, when c is small, there are many other best invariant estimates.

4.7.3. The parameter θ is a location parameter for the distribution of X, so the Pitman estimate is given by (5.58):

$$\hat{\theta} = d_0(\mathbf{X}) = \frac{\int \theta f(X_1 - \theta, \ldots, X_n - \theta)\, d\theta}{\int f(X_1 - \theta, \ldots, X_n - \theta)\, d\theta} = \frac{\int_{-\infty}^{\min(X_i)} (\sqrt{2/\pi})^n \exp\{-\frac{1}{2}\sum (X_i - \theta)^2\}\, d\theta}{\int_{-\infty}^{\min(X_i)} \theta(\sqrt{2/\pi})^n \exp\{-\frac{1}{2}\sum (X_i - \theta)^2\}\, d\theta}$$

$$= \frac{\int_{-\infty}^{\min(X_i)} \theta \exp\{-\frac{n}{2}\sum (\theta - \overline{X})^2\}\, d\theta}{\int_{-\infty}^{\min(X_i)} \exp\{-\frac{n}{2}\sum (\theta - \overline{X})^2\}\, d\theta}.$$

Make the change of variable $y = \theta - \overline{X}$ for θ.

$$\hat{\theta} = \overline{X} + \frac{\int_{-\infty}^{\min(X_i)-\overline{X}} ye^{-ny^2/2}\,dy}{\int_{-\infty}^{\min(X_i)-\overline{X}} e^{-ny^2/2}\,dy} = \overline{X} - \frac{(1/n)\exp\{-n(\min(X_i)-\overline{X})^2/2\}}{\sqrt{2\pi/n}\Phi(\sqrt{n}(\min(X_i)-\overline{X}))}$$
$$= \overline{X} - \frac{\exp\{-n(\min(X_i)-\overline{X})^2/2\}}{\sqrt{2\pi n}\Phi(\sqrt{n}(\min(X_i)-\overline{X}))}$$

where $\min(X_i) = \min\{X_1,\ldots,X_n\}$, $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$, and $\Phi(\cdot)$ is the distribution function of $\mathcal{N}(0,1)$.

4.7.4. Consider the distribution of $X_1, Y_2, \ldots, Y_n$, where $Y_2 = X_2/X_1, Y_3 = X_3/X_1, \ldots, Y_n = X_n/X_1$. The distribution of $\mathbf{Y} = (Y_2,\ldots,Y_n)$ does not depend on θ since $Y_j = (X_j/\theta)/(X_1/\theta)$, and the distribution of X_1 given $\mathbf{Y}$ has θ as a location parameter. Since the loss function is a function only of a/θ, $L(\theta,a) = W(a/\theta)$, the best invariant estimate of θ for the conditional problem given $\mathbf{Y}$ is

$$d_0(\mathbf{X}) = \frac{X_1}{b_0(\mathbf{Y})}$$

where $b_0(\mathbf{Y})$ minimizes $\mathrm{E}_1(W(X_1/b)|\mathbf{Y})$. ($\mathrm{E}_1$ stands for the expectation when $\theta = 1$.) For the special case $W(a/\theta) = (a-\theta)^2/\theta^2$, we have that $b_0(\mathbf{Y})$ is that value of b that minimizes the expected weighted squared error, $\mathrm{E}_1[(X_1-b)^2/b^2|\mathbf{Y}]$. Thus,

$$b_0(\mathbf{Y}) = \frac{\mathrm{E}_1(X_1^2|\mathbf{Y})}{\mathrm{E}_1(X_1|\mathbf{Y})}.$$

The density of $X_1, Y_2, \ldots, Y_n$ when $\theta = 1$ is

$$f_{X_1,Y_2,\ldots,Y_n}(x_1,y_2,\ldots,y_n) = f(x_1, y_2x_1, \ldots, y_nx_1)x_1^{n-1}$$

so that the conditional distribution of X_1 given $\mathbf{Y}$ when $\theta = 1$ is

$$f_{X_1|Y_2=y_2,\ldots,Y_n=y_n}(x_1|\theta=1) = \frac{f(x_1,y_2x_1,\ldots,y_nx_1)x_1^{n-1}}{\int f(x_1,y_2x_1,\ldots,y_nx_1)x_1^{n-1}\,dx_1}.$$

Hence, the best invariant estimate is

$$d_0(\mathbf{X}) = \frac{X_1}{b_0(\mathbf{Y})} = \frac{X_1\mathrm{E}_1(X_1|\mathbf{Y})}{\mathrm{E}(X_1^2|\mathbf{Y})} = \frac{X_1\int_0^\infty x_1 f(x_1,Y_2x_1,\ldots,Y_nx_1)x_1^{n-1}\,dx_1}{\int_0^\infty x_1^2 f(x_1,Y_2x_1,\ldots,Y_nx_1)x_1^{n-1}\,dx_1}$$
$$= \frac{\int_0^\infty \theta^{-(n+2)} f(X_1/\theta,\ldots,X_n/\theta)\,d\theta}{\int_0^\infty \theta^{-(n+3)} f(X_1/\theta,\ldots,X_n/\theta)\,d\theta}$$

where we have made the change of variable of integration, $\theta = X_1/x_1$.

4.7.5. The conditions of problem 4.7.4 are satisfied with

$$f(x_1,\ldots,x_n) = \prod_{i=1}^n \mathrm{I}(1 \le x_i \le 2),$$

so the best invariant rule is

$$d_0(\mathbf{X}) = \frac{\int_0^\infty \theta^{-(n+2)} f(X_1/\theta,\ldots,X_n/\theta)\,d\theta}{\int_0^\infty \theta^{-(n+3)} f(X_1/\theta,\ldots,X_n/\theta)\,d\theta}.$$

If $U = \min(X_i)$ and $V = \max(X_i)$, then

$$f(x_1/\theta,\ldots,x_n/\theta) = \mathrm{I}(V/2 \le \theta \le U).$$

Hence, the numerator of $d_0(\mathbf{X})$ is

$$\int_0^\infty \theta^{-(n+2)} f(X_1/\theta, \ldots, X_n/\theta)\, d\theta = \int_{v/2}^U \theta^{-(n+2)}\, d\theta = \frac{1}{n+1}((V/2)^{-(n+1)} - U^{-(n+1)})$$

and similarly the denominator is

$$\frac{1}{n+1}((V/2)^{-(n+1)} - U^{-(n+1)})$$

so that the best invariant rule is

$$d_0(\mathbf{X}) = \frac{(n+2)[(V/2)^{-(n+1)} - U^{-(n+1)}]}{(n+1)[(V/2)^{-(n+2)} - U^{-(n+2)}]}.$$

4.7.6. Suppose $X_1, \ldots, X_n$ has density (4.48) (note the correction), and suppose that $u(\mathbf{x})$ is an arbitrary but fixed invariant function (for example, $u(\mathbf{X}) = \overline{X}$). Then $u(x_1 - c, \ldots, x_n - c) = u(x_1, \ldots, x_n) - c$ identically in $x_1, \ldots, x_n$ and c. If we replace c by x_1, we find that $u(0, x_2 - x_1, \ldots, x_n - x_1) = u(x_1, \ldots, x_n) - x_1$, or, $u(\mathbf{x}) = x_1 - u(0, y_2, \ldots, y_n)$, where $y_i = x_i - x_1$ for $i = 2, \ldots, n$. Thus, the invariant rule (4.49) can be written as $u(\mathbf{X})$ plus some function of the vector of differences, $\mathbf{Y}$, say $d_0(\mathbf{X}) = u(\mathbf{X}) + b_0'(\mathbf{Y})$. The best invariant rule is found with $b_0'(\mathbf{Y})$ as that number b that minimizes $\mathrm{E}_0(l(u(\mathbf{X}) - b)|\mathbf{Y})$.