Solutions to the Exercises of Section 3.6.

3.6.1. The distribution function of $T = \max_{1\le i\le n} X_i$ is

$$\begin{aligned} F_T(t|\theta) &= \mathrm{P}_\theta(\max_{1\le i\le n} X_i \le t) = \mathrm{P}_\theta(X_1 \le t, \ldots, X_n \le t) \\ &= \mathrm{P}_\theta(X_1 \le t)\cdots \mathrm{P}_\theta(X_n \le t) \qquad \text{(since independent)} \\ &= \mathrm{P}_\theta(X \le t)^n \qquad \text{(since identically distributed)} \\ &= \begin{cases} 0 & \text{if } t < 0 \\ (t/\theta)^n & \text{if } 0 \le t < \theta \\ 1 & \text{if } t > 1 \end{cases} \end{aligned}$$

The density is the derivative of this function with respect to t,

$$f_T(t|\theta) = \frac{nt^{n-1}}{\theta^n} I_{(0,\theta)}(t).$$

3.6.2. From 3.6.1 with $\theta = 1$, we have

$$\mathrm{E} \max_{1\le i\le n} X_i = \int_0^1 t\, nt^{n-1}\, dt = \frac{n}{n+1} t^{n+1}|_0^1 = \frac{n}{n+1}.$$

The distribution of these X_i is symmetric about $1/2$ so

$$\mathrm{E} \min_{1\le i\le n} X_i = 1 - \mathrm{E} \max_{1\le i\le n} X_i = 1 - \frac{n}{n+1} = \frac{1}{n+1}.$$

When $X_1, \ldots, X_n$ is a sample from $\mathcal{U}(\theta - 1/2, \theta + 1/2)$, the distribution of $\max_{1\le i\le n} X_i - \min_{1\le i\le n} X_i$ does not depend on θ (since θ is a location parameter and $\max_{1\le i\le n} X_i - \min_{1\le i\le n} X_i$ is a function of the differences), so

$$\mathrm{E}_\theta(\max_{1\le i\le n} X_i - \min_{1\le i\le n} X_i) = \mathrm{E}_{1/2}(\max_{1\le i\le n} X_i - \min_{1\le i\le n} X_i) = \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}.$$

3.6.3. Since $\sum_1^n X_i \in \mathcal{N}(n\mu, n\sigma^2)$, we have $\mathrm{E}(\sum_1^n X_i)^2 = n\sigma^2 + (n\mu)^2$. On the other hand, $\mathrm{E}\sum_1^n X_i^2 = n\mathrm{E}X^2 = n(\sigma^2 + \mu^2)$. Hence $\mathrm{E}(2(\sum_1^n X_i)^2 - (n+1)\sum_1^n X_i^2) = 2n\sigma^2 + 2n^2\mu^2 - (n+1)n\sigma^2 - (n+1)n\mu^2 = n(n-1)(\mu^2 - \sigma^2)$.

3.6.4. The joint density of T_1 and T_2 is of the form

$$f_{T_1,T_2}(t_1, t_2|\pi_1, \pi_2) = c(\pi_1, \pi_2)h(t_1, t_2)\mathrm{I}(\pi_1 < t_1 < t_2 < \pi_2).$$

where we are assuming that

$$c(\pi_1, \pi_2)^{-1} = \int_{\pi_1}^{\pi_2} \int_{\pi_1}^{t_2} h(t_1, t_2)\, dt_1 dt_2$$

exists and is finite for all $\pi_1 < \pi_2$. To show completeness, suppose that $\mathrm{E}_{\pi_1,\pi_2} g(T_1, T_2) = 0$ for all $\pi_1 < \pi_2$, or equivalently, that

$$\phi(\pi_1, \pi_2) = \int_{\pi_1}^{\pi_2} \int_{\pi_1}^{t_2} g(t_1, t_2)h(t_1, t_2)\, dt_1 dt_2 = 0$$

for all $\pi_1 < \pi_2$. If we assume this exists as a Riemann integral, then the integrand, $g(t_1, t_2)h(t_1, t_2)$, is continuous at almost all points of the plane. Then taking the partial derivatives with respect to π_1 and π_2 and using the Fundamental Theorem of Calculus, we find that $g(\pi_1, \pi_2)h(\pi_1, \pi_2) = 0$ for all points (π_1, π_2) at which the integrand is continuous, namely at almost all points of the plane. Hence, $g(t_1, t_2)$ is zero except on a set of points of zero probability, showing completeness.

3.6.5. If $X \in \mathcal{N}(0, \sigma^2)$, then X is not complete since $\mathrm{E}_\sigma X = 0$ for all σ. From the Factorization Theorem, $T = X^2$ is a sufficient statistic for σ. The density of T is

$$f_T(t|\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{t}{2\sigma^2}\} \frac{1}{\sqrt{t}}.$$

This is a one-parameter exponential family, (in fact, a gamma family), and so T is complete.

3.6.6. For an arbitrary function $g(x)$, the expectation of $g(X)$ may be written as a power series in θ as follows.

$$\begin{aligned} \mathrm{E}g(X) &= \theta g(-1) + (1-\theta^2) \sum_{x=0}^{\infty} \theta^x g(x) \\ &= \theta g(-1) + \sum_{x=0}^{\infty} (\theta^x - 2\theta^{x+1} + \theta^{x+2}) g(x) \\ &= g(0) + \theta(g(-1) - 2g(0) + g(1)) + \theta^2(g(0) - 2g(1) + g(2)) + \cdots \end{aligned}$$

If this is identically zero for $0 < \theta < 1$, then each coefficient of the power series must be zero. This gives immediately that $g(0) = 0$, and writing each of the other coefficients in terms of $g(1)$ we find

$$\begin{aligned} g(-1) - 2g(0) + g(1) &= 0 \qquad & g(-1) &= -g(1) \\ g(0) - 2g(1) + g(2) &= 0 \qquad & g(2) &= 2g(1) \\ g(1) - 2g(2) + g(3) &= 0 \qquad & g(3) &= 3g(1) \\ g(2) - 2g(3) + g(4) &= 0 \qquad & g(4) &= 4g(1) \\ & \cdots \end{aligned}$$

If $g(x)$ is assumed bounded, then $g(1)$ must be zero, and hence all other $g(x)$ must be zero. This shows that the distribution of X is boundedly complete. On the other hand, the function $g(x) = x$ satisfies all the equalities and so $\mathrm{E}_\theta X = 0$ for all θ showing that the distribution of X is not complete.