Solutions to the Exercises of Section 2.3.

2.3.1. Suppose that δ_0 is Bayes with respect to τ,

$$r(\tau, \delta_0) = \inf_{\delta} r(\tau, \delta),$$

and that that it is unique Bayes up to equivalence, i.e. any other rule Bayes with to τ has the same risk function as δ_0. We are to show that δ_0 is admissible. We proceed by contradiction. Suppose that δ_0 is not admissible. Then there exists a rule δ_1 such that

$$R(\theta, \delta_1) \leq R(\theta, \delta_0) \quad \text{for all } \theta,$$

with strict inequality for some θ. Taking expectation with respect to τ on both sides of this inequality gives

$$r(\tau, \delta_1) \leq r(\tau, \delta_0) = \inf_{\delta} r(\tau, \delta),$$

which shows that δ_1 is also Bayes with respect to τ. But the risk function of δ_1 differs from the risk function of δ_0, which shows that δ_0 is not unique Bayes up to equivalence, a contradiction that completes the proof.

2.3.2. Suppose that the points $\theta_1, \theta_2, \ldots$ are in the support of a distribution τ on the real line and suppose $\theta_n \to \theta$ as $n \to \infty$. We are to show that θ is in the support of τ. Let $\epsilon > 0$ We are to show $\tau(\theta - \epsilon, \theta + \epsilon) > 0$. But since $\theta_n \to \theta$, there exists an integer k such that θ_k is in the interval $(\theta - \epsilon/2, \theta + \epsilon/2)$, and since θ_k is in the support of τ, we have $\tau(\theta_k - \epsilon/2, \theta_k + \epsilon/2) > 0$. The result now follows from $\tau(\theta - \epsilon, \theta + \epsilon) > \tau(\theta_k - \epsilon/2, \theta_k + \epsilon/2) > 0$.

2.3.3. Let δ_0 be ϵ-Bayes with respect to τ for a fixed $\epsilon \geq 0$, i.e.

$$r(\tau, \delta_0) \leq \inf_{\delta} r(\tau, \delta) + \epsilon.$$

Suppose that δ_0 is not ϵ-admissible; then there exists a rule δ_1 such that $R(\theta, \delta_1) < R(\theta, \delta_0) - \epsilon$ for all $\theta \in \Theta$. Then,

$$r(\tau, \delta_1) = \mathrm{E}\, R(\theta, \delta_1) < \mathrm{E}\, R(\theta, \delta_0) - \epsilon = r(\tau, \delta_0) - \epsilon,$$

where the expectation is taken over θ using the distribution τ. This contradicts the assumption that δ_0 is ϵ-Bayes. Note that this result is true for $\epsilon = 0$ as well, but that 0-admissibility is a weaker property than admissibility.