Large Sample Theory
Ferguson

Exercises, Section 20, Asymptotic Efficiency.

1. (Sometimes twice is useful.) When the error in Newton's method gets small enough, the error goes down quadratically; that is, if the error at one stage is ϵ, the error at the next stage will be bounded by $c\epsilon^2$ for some constant c independent of ϵ. Suppose the conditions of Theorem 18 are satisfied and suppose $\tilde{\theta}_n$ is an estimate that is only $O(n^{-1/4})$-consistent (i.e. $n^{-1/4}(\tilde{\theta}_n - \theta)$ is bounded in probability). Let $\tilde{\theta}_n^{(1)}$ be the improvement of $\tilde{\theta}_n$ given by one iteration of Newton's method. Show that $\tilde{\theta}_n^{(1)} - \theta$ is $O(n^{-1/2})$. Conclude that two iterations of Newton's method, applied to $\tilde{\theta}_n$ gives a fully efficient estimate.

2. Let $X_1, \ldots, X_n$ be a sample from the beta distribution, $\mathcal{B}e(\theta, 1)$, with density, $f(x|\theta) = \theta x^{\theta-1}\,\mathrm{I}(0 < x < 1)$.
(a) What is the asymptotic distribution of $\tilde{\theta}_n$, the estimate of θ given by the method of moments?
(b) What is $\tilde{\theta}_n^{(1)}$, the improvement of $\tilde{\theta}_n$ given by one iteration of Newton's method?
(c) What is the asymptotic efficiency of $\tilde{\theta}_n$ relative to $\tilde{\theta}_n^{(1)}$?

3. Let μ be the median of the beta distribution with density $f(x|\theta) = \theta x^{\theta-1}\,\mathrm{I}(0 < x < 1)$, and let $\hat{\mu}_n$ represent the median of a sample of size n from this distribution.
(a) Find μ as a function of θ.
(b) Find the asymptotic distribution of $\sqrt{n}(\hat{\mu}_n - \mu)$ as $n \to \infty$.
(c) Find the Cramér-Rao lower bound for the variance of an unbiased estimate of μ.
(d) What is the asymptotic efficiency of $\hat{\mu}_n$ as an estimate of μ relative to the maximum likelihood estimate.

4. Let $\Theta = (0, \infty)$ and let $\mathcal{F}$ be the class of all one-parameter families of distributions $F(x|\theta)$ for $\theta \in \Theta$ with mean θ and variance θ for which the conditions of the information inequality with $g(x) = x$ are satisfied. (For example $\mathcal{G}(\theta, 1)$.) Let $\mathcal{I}_F(\theta)$ denote Fisher information.
(a) Show $\mathcal{I}_F(\theta) \geq 1/\theta$ for all $F \in \mathcal{F}$.
(b) Show that out of all $F \in \mathcal{F}$, $\mathcal{I}_F(\theta)$ is smallest if F is the Poisson distribution with mean θ.

5. Let $X_1, \ldots, X_n$ be a sample from a mixture of exponential distributions:

$$f(x|\theta) = (1-\theta)\,e^{-x} + \theta\,(1/2)e^{-x/2} \qquad \text{for} \quad x > 0$$

where $0 \leq \theta \leq 1$.
(a) What is the estimate of θ given by the method of moments?
(b) Show how to improve this estimate by one iteration of Newton's method applied to the likelihood equation (the method of scoring).

6. Two students were requested to perform an experiment to obtain a sample of size n from an exponential distribution with an unknown mean θ. One student did as requested

and recorded the values as $X_1, \ldots, X_n$. The other student independently collected a sample but only recorded the integer parts of the observations, which were recorded as $Y_1, \ldots, Y_n$. (Each Y_i is the biggest integer less than or equal to the corresponding observation.)

(a) Show that $Y_1, \ldots, Y_n$ is a sample from a geometric distribution with some probability p of success, $f(y) = (1-p)p^y$ for $y = 0, 1, 2, \ldots$, and find p as a function of θ.

(b) The main problem is how to combine the two samples to get a good (i.e. asymptotically efficient) estimate of θ. Find the likelihood equations. Use the method of scoring with some reasonable preliminary estimate to get an asymptotically efficient estimate of θ.

(c) What is the asymptotic variance of your estimate?