**Large Sample Theory**
Ferguson

**Exercises, Section 11, Stationary $m$-Dependent Sequences.**

1. **A Moving Average Process.** Suppose $X_1, X_2, \ldots$ are i.i.d. unobservable (latent) random variables with an unknown distribution having finite mean $\mu$ and variance $\sigma_x^2$. The observations are $Y_t = \beta_0 X_t + \beta_1 X_{t+1} + \cdots + \beta_m X_{t+m}$ for $t = 1, 2, \ldots$, where the $\beta_i$ are constants. Such a process $Y_1, Y_2, \ldots$ is said to be a moving average process of order $m$. Find the asymptotic distribution of $\overline{Y}_n$. What happens if $\beta_0 + \beta_1 + \cdots + \beta_m = 0$?

2. Usually a normalizing condition is placed on a moving average process, either $\sum \beta_i = 1$ or $\beta_0 = 1$ or $\sigma_x^2 = 1$. Take $m = 1$, $\mu_x = 0$ and $\sigma_x^2 = 1$ in Problem 1 above. How would you go about finding consistent estimators of $\beta_0$ and $\beta_1$? Be careful. The $\beta_i$ may not be identifiable without further restrictions (that may be made without loss of generality). In particular, what should the parameter space be?

3. **An Auto-Regressive Process.** Consider Exercise 11.7 with $z_j = 0$ for $j < 0$ and $z_j = \beta^j$ for $j \geq 0$, where $|\beta| < 1$. Then $Y_t = \sum_0^\infty \beta^j X_{t-j} = \beta Y_{t-1} + X_t$ for all $t$, so this is just the auto-regressive model investigated in Exercises 4.2 and 6.3. Use Exercise 11.7 to find the asymptotic distribution of $\overline{Y}_n = n^{-1} \sum_1^n Y_j$.

4. **An ARMA (Auto-Regressive Moving Average) Model.** We may combine the auto-regressive process of Exercise 3 with a moving average process of Exercise 1 to obtain an ARMA model, the simplest of which is $Y_t = \beta_0 Y_{t-1} + \beta_1 X_{t-1} + X_t$, where $|\beta_0| < 1$. Assume stationarity of the $Y_t$ and use Exercise 11.7 to find the asymptotic distribution of $\overline{Y}_n$.

5. (a) Extend Theorem 11 to the multivariate case. (The simplest way is to use Theorem 11 in one dimension together with the "Cramér-Wold device", namely, Exercise 2 of Section 3.) Be careful. Remember that for vectors, $\mathrm{Cov}(\boldsymbol{X}, \boldsymbol{Y})$, defined as $\mathrm{E}(\boldsymbol{X}\boldsymbol{Y}^T) - \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{Y})^T$, is not necessarily the same as $\mathrm{Cov}(\boldsymbol{Y}, \boldsymbol{X})$.

(b) Apply the result of part (a), to the badminton problem, Exercise 3 of Section 11. Find the joint asymptotic distribution of the number of successes, $R_n = \sum_1^n X_i$, and the number of points scored by time $n$, $S_n = \sum_1^n X_{i-1} X_i$.

6. Let $X_0, X_1, X_2, \ldots$ be a sequence of independent Bernoulli trials with probability $p$ of success. Each success is worth one point but two successes in a row gives an extra point. So the number of points received on the $n$th trial is

$$Y_n = \begin{cases} 2 & \text{if } X_n \text{ and } X_{n-1} \text{ are successes} \\ 1 & \text{if } X_n \text{ is a success and } X_{n-1} \text{ is a failure} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the mean and the variance of $Y_n$.
(b) Find the asymptotic distribution of $S_n = \sum_1^n Y_i$ (properly normalized).