Large Sample Theory
Ferguson

Exercises, Section 7, Functions of the Sample Moments.

1. Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean $\mu$ and variance $\sigma^2$. Find the asymptotic distribution of $R_n = \sum_{i=1}^{n} X_{2i-1}/\sum_{i=1}^{n} X_{2i}$ for (a) $\mu \neq 0$, and for (b) $\mu = 0$.

2. Professor Bliss has at hand a large sample $X_1, \ldots, X_n$, from the double exponential distribution with density $f(x) = (1/(2\tau))e^{-(|x| - \mu)/\tau}$, having mean $\mu$ and mean deviation $E|X - \mu| = \tau$. He knows enough to estimate $\mu$ by the sample median, $m_n$, and he knows he should use $(1/n)\sum |X_i - m_n|$ to estimate the mean deviation (these are the MLE’s), or $(1/n)\sum (X_i - m_n)^2$ to estimate the variance, $\sigma^2 = 2\tau^2$, but he doesn’t quite know what the sampling distribution might be. He decides instead to use the sample variance, $(1/n)\sum (X_i - \bar{X}_n)^2$, to estimate $\sigma^2$, and to get confidence intervals for $\sigma^2$ using the chi-square tables. How well is Professor Bliss doing in his confidence intervals for $\sigma^2$? (You may assume $n$ large.)

3. Let $X$ have the Poisson distribution, $P(\lambda)$. We know that $(X - \lambda)/\sqrt{\lambda} \overset{L}{\to} N(0,1)$ as $\lambda \to \infty$, and we say $X \sim N(\lambda, \lambda)$ for large $\lambda$.
   (a) Show $\log(X) \sim N(\log(\lambda), \lambda^{-1})$ for large $\lambda$.
   (b) Show $X^2 \sim N(\lambda^2, 4\lambda^3)$ for large $\lambda$.
   (c) Is it true that $e^X \sim N(e^\lambda, \text{something})$ for large $\lambda$?

4. Let $X_1, \ldots, X_n$ be a sample from the geometric distribution with mass function, $P(X = x) = (1 - \theta)\theta^x$ for $x = 0, 1, \ldots$, where $0 < \theta < 1$ is a success probability. Let $S_n = \sum_1^n X_i$ denote the total number of successes, and $T_n = \sum_1^n I(X_i > 0)$ denote the number of trials that had at least one success.
   (a) Find the joint asymptotic distribution of $(S_n, T_n)$.
   (b) Find the joint asymptotic distribution of $(U_n, V_n)$, where $U_n = S_n/T_n$ and $V_n = n - T_n$.

5. To estimate a parameter, $\theta^2$, you are given the choice of the following two possibilities: (1) the estimate $\overline{X}_n^2$, based on a sample, $X_1, \ldots, X_n$ from the gamma distribution, $G(\theta, 1)$, and (2) the estimate $\sum S_n$, based on a sample, $Y_1, \ldots, Y_n$ from the gamma distribution, $G(\theta^2, 1)$. If $n$ is large, which would you choose? (The answer depends on $\theta$.)

6. If $\sqrt{n}(\overline{X}_n - \theta) \overset{L}{\to} N(0, \sigma^2)$ as $n \to \infty$, what is the asymptotic distribution of $|\overline{X}_n|$? (Consider the cases $\theta = 0$ and $\theta \neq 0$ separately.)

7. Let $X_1, \ldots, X_n$ be a sample from $N(\theta, \sigma^2)$ with $\sigma^2$ known. For a fixed number $a$, let $p = P(X_i > a) = 1 - \Phi((a - \theta)/\sigma) = \Phi((\theta - a)/\sigma)$. The maximum likelihood estimate of $p$ is therefore $\hat{p}_n = \Phi((\overline{X}_n - a)/\sigma)$. Find the asymptotic distribution of $\sqrt{n}(\hat{p}_n - p)$.

8. Let $X_1, \ldots, X_n$ be i.i.d. with mean zero and positive finite sixth moment. Let $\mu_k = E(X^k)$ denote the population moments and $m_k = (1/n)\sum X_i^k$ denote the sample moments. Then $m_2$ is a reasonable estimate of $\mu_2$ and has asymptotic distribution

$$\sqrt{n}(m_2 - \mu_2) \overset{L}{\to} N(0, \mu_4 - \mu^2_2).$$
Show that the estimate of $\mu_2$ given by

$$\hat{\sigma}^2 = m_2 - \frac{m_1 m_3}{m_2}$$

has an asymptotic normal distribution,

$$\sqrt{n}(\hat{\sigma}^2 - \mu_2) \xrightarrow{L} \mathcal{N}(0, \tau^2).$$

with some asymptotic variance $\tau^2$. Find $\tau^2$ and show that $\tau^2 \leq \mu_4 - \mu_2^2$, with equality if and only if $\mu_3 = 0$. Note that for two-point distributions, $\tau^2 = 0$.

9. (a) Suppose $\sqrt{n}(Z_n - \sigma^2) \xrightarrow{L} \mathcal{N}(0, 2\sigma^4)$, where $\sigma > 0$. Find the asymptotic distribution of $\sqrt{n}(\sqrt{Z_n} - \sigma)$.

(b) Find the approximation given by the second order Taylor expansion to the asymptotic distribution of $\sqrt{n}(\sqrt{Z_n} - \sigma)$.

(c) Take $n = 10$, $\sigma = 1$ and suppose the original distribution of $\sqrt{n}(Z_n - \sigma)$ is exactly normal. Find the exact probability, $P(\sqrt{n}(\sqrt{Z_n} - \sigma) > .5)$, and compare it to the approximations given by (a) and (b).

(d) Suppose the distribution of $Z_n$ is not normal but instead that $nZ_n/\sigma^2$ is exactly $\chi^2_n$, as it would be if $Z_n$ were the sample variance of a sample of size $n + 1$ from a normal distribution with variance $\sigma^2$ (i.e. $Z_n = (1/n) \sum_{i=1}^{n+1} (X_i - \overline{X_{n+1}})^2 = \sigma^2$). Now find the exact probability $P(\sqrt{n}(\sqrt{Z_n} - \sigma) > .5)$ for $n = 10$ and $\sigma = 1$, and compare it to the approximations given by (a) using the Edgeworth expansions. (Note that Table 1 has fortuitously been constructed for the normalized $\chi^2_{10}$ distribution.)

10. For convenience, Cramér’s Theorem has been stated assuming $g'(x)$ is continuous in a neighborhood of $\mu$. It also holds under the weaker assumption that $g'(x)$ exists at $\mu$ in the sense that

$$\frac{g(x) - g(\mu)}{x - \mu} \to g'(\mu)$$

as $x \to \mu, x \neq \mu$. Show this in one dimension:

**Theorem.** Let $g(x)$ be defined in a neighborhood of $\mu$ and assume that $g'(x)$ exists at $\mu$. If $b_n(X_n - \mu) \xrightarrow{L} X$, where $b_n$ is a sequence of numbers tending to infinity. Then $b_n(g(X_n) - g(\mu)) \xrightarrow{L} g'(\mu)X$. 