

Large Sample Theory

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Exercises, Section 7, Functions of the Sample Moments.

1. Let X_1, X_2, \dots be i.i.d. random variables with mean μ and variance σ^2 . Find the asymptotic distribution of $R_n = \sum_{i=1}^n X_{2i-1} / \sum_{i=1}^n X_{2i}$ for (a) $\mu \neq 0$, and for (b) $\mu = 0$.

2. Professor Bliss has at hand a large sample X_1, \dots, X_n , from the double exponential distribution with density $f(x) = (1/(2\tau))e^{-|x-\mu|/\tau}$, having mean μ and mean deviation $E|X - \mu| = \tau$. He knows enough to estimate μ by the sample median, m_n , and he knows he should use $(1/n) \sum |X_i - m_n|$ to estimate the mean deviation (these are the MLE's), or $(1/n) \sum (X_i - m_n)^2$ to estimate the variance, $\sigma^2 = 2\tau^2$, but he doesn't quite know what the sampling distribution might be. He decides instead to use the sample variance, $(1/n) \sum (X_i - \bar{X}_n)^2$, to estimate σ^2 , and to get confidence intervals for σ^2 using the chi-square tables. How well is Professor Bliss doing in his confidence intervals for σ^2 ? (You may assume n large.)

3. Let X have the Poisson distribution, $\mathcal{P}(\lambda)$. We know that $(X - \lambda)/\sqrt{\lambda} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$ as $\lambda \rightarrow \infty$, and we say $X \sim \mathcal{N}(\lambda, \lambda)$ for large λ .

(a) Show $\log(X) \sim \mathcal{N}(\log(\lambda), \lambda^{-1})$ for large λ .

(b) Show $X^2 \sim \mathcal{N}(\lambda^2, 4\lambda^3)$ for large λ .

(c) Is it true that $e^X \sim \mathcal{N}(e^\lambda, \text{something})$ for large λ ?

4. Let X_1, \dots, X_n be a sample from the geometric distribution with mass function, $P(X = x) = (1 - \theta)\theta^x$ for $x = 0, 1, \dots$, where $0 < \theta < 1$ is a success probability. Let $S_n = \sum_1^n X_i$ denote the total number of successes, and $T_n = \sum_1^n I(X_i > 0)$ denote the number of trials that had at least one success.

(a) Find the joint asymptotic distribution of (S_n, T_n) .

(b) Find the joint asymptotic distribution of (U_n, V_n) , where $U_n = S_n/T_n$ and $V_n = n - T_n$.

5. To estimate a parameter, θ^2 , you are given the choice of the following two possibilities: (1) the estimate \bar{X}_n^2 , based on a sample, X_1, \dots, X_n from the gamma distribution, $\mathcal{G}(\theta, 1)$, and (2) the estimate \bar{Y}_n , based on a sample, Y_1, \dots, Y_n from the gamma distribution, $\mathcal{G}(\theta^2, 1)$. If n is large, which would you choose? (The answer depends on θ .)

6. If $\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$ as $n \rightarrow \infty$, what is the asymptotic distribution of $|\bar{X}_n|$? (Consider the cases $\theta = 0$ and $\theta \neq 0$ separately.)

7. Let X_1, \dots, X_n be a sample from $\mathcal{N}(\theta, \sigma^2)$ with σ^2 known. For a fixed number a , let $p = P(X_i > a) = 1 - \Phi((a - \theta)/\sigma) = \Phi((\theta - a)/\sigma)$. The maximum likelihood estimate of p is therefore $\hat{p}_n = \Phi((\bar{X}_n - a)/\sigma)$. Find the asymptotic distribution of $\sqrt{n}(\hat{p}_n - p)$.

8. Let X_1, \dots, X_n be i.i.d. with mean zero and positive finite sixth moment. Let $\mu_k = E(X^k)$ denote the population moments and $m_k = (1/n) \sum_1^n X_i^k$ denote the sample moments. Then m_2 is a reasonable estimate of μ_2 and has asymptotic distribution

$$\sqrt{n}(m_2 - \mu_2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \mu_4 - \mu_2^2).$$

Show that the estimate of μ_2 given by

$$\hat{\sigma}^2 = m_2 - \frac{m_1 m_3}{m_2}$$

has an asymptotic normal distribution,

$$\sqrt{n}(\hat{\sigma}^2 - \mu_2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tau^2).$$

with some asymptotic variance τ^2 . Find τ^2 and show that $\tau^2 \leq \mu_4 - \mu_2^2$, with equality if and only if $\mu_3 = 0$. Note that for two-point distributions, $\tau^2 = 0$.

9. (a) Suppose $\sqrt{n}(Z_n - \sigma^2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2\sigma^4)$, where $\sigma > 0$. Find the asymptotic distribution of $\sqrt{n}(\sqrt{Z_n} - \sigma)$.

(b) Find the approximation given by the second order Taylor expansion to the asymptotic distribution of $\sqrt{n}(\sqrt{Z_n} - \sigma)$.

(c) Take $n = 10$, $\sigma = 1$ and suppose the original distribution of $\sqrt{n}(Z_n - \sigma^2)$ is exactly normal. Find the exact probability, $P(\sqrt{n}(\sqrt{Z_n} - \sigma) > .5)$, and compare it to the approximations given by (a) and (b).

(d) Suppose the distribution of Z_n is not normal but instead that nZ_n/σ^2 is exactly χ_n^2 , as it would be if Z_n were the sample variance of a sample of size $n + 1$ from a normal distribution with variance σ^2 (i.e. $Z_n = (1/n) \sum_1^{n+1} (X_i - \bar{X}_{n+1})^2 = s_x^2$). Now find the exact probability $P(\sqrt{n}(\sqrt{Z_n} - \sigma) > .5)$ for $n = 10$ and $\sigma = 1$, and compare it to the approximations given by (a) using the Edgeworth expansions. (Note that Table 1 has fortuitously been constructed for the normalized χ_{10}^2 distribution.)

10. For convenience, Cramér's Theorem has been stated assuming $g'(x)$ is continuous in a neighborhood of μ . It also holds under the weaker assumption that $g'(x)$ exists at μ in the sense that $\frac{g(x) - g(\mu)}{x - \mu} \rightarrow g'(\mu)$ as $x \rightarrow \mu$, $x \neq \mu$. Show this in one dimension.

Theorem. Let $g(x)$ be defined in a neighborhood of μ and assume that $g'(x)$ exists at μ . If $\sqrt{n}(X_n - \mu) \xrightarrow{\mathcal{L}} X$, then $\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{\mathcal{L}} g'(\mu)X$.