Large Sample Theory
Ferguson

Exercises, Section 4, Laws of Large Numbers.

1. (a) Suppose $X_1, X_2, \ldots, X_n$ are independent Poisson random variables with means, $\mathrm{E}X_i = \lambda z_i$, where the z_i are known constants, and $\lambda > 0$ is an unknown parameter. The ordinary least squares estimate of λ, $\hat{\lambda}_{LS}$, is the value of λ that minimizes $S^2 = \sum (X_i - \lambda z_i)^2$. The weighted least squares estimate of λ, λ_W, with weights proportional to the inverse of the variance, is the value of λ that minimizes $S_W^2 = \sum (X_i - \lambda z_i)^2 / z_i$. Find these estimates.

(b) Under what conditions on the z_i is it true that $\hat{\lambda}_{LS} \xrightarrow{\text{q.m.}} \lambda$ as $n \to \infty$?

(c) Under what conditions on the z_i is it true that $\hat{\lambda}_W \xrightarrow{\text{q.m.}} \lambda$ as $n \to \infty$?

(d) Give an example where the weighted least squares estimate of λ is consistent in quadratic mean, and the ordinary least squares estimate is not.

2. Let $X, X_1, X_2, \ldots$ be i.i.d. with probability mass function, $\mathrm{P}(X = j) = \mathrm{P}(X = -j) = c/(j^2 \log j)$ for $j = 3, 4, \ldots$, where $c = (2 \sum_3^\infty 1/(j^2 \log j))^{-1}$ is the normalizing constant. Show that $\overline{X}_n \xrightarrow{\mathrm{P}} 0$.

This gives an example of a distribution that obeys the weak law of large numbers even though $\mathrm{E}(X)$ does not exist. Note that Theorem 4(c) implies that $\overline{X}_n$ does not converge a.s. to 0. Show $\overline{X}_n \xrightarrow{\mathrm{P}} 0$ in three steps.

(a) Let $Y_{n,k} = X_k \mathrm{I}(|X_k| \le n)$ for $k = 1, \ldots, n$. Show that $\overline{Y}_n \xrightarrow{\text{qm}} 0$ so that $\overline{Y}_n \xrightarrow{\mathrm{P}} 0$.

(b) Show that $\mathrm{P}(\overline{X}_n \neq \overline{Y}_n) \le \sum_{k=1}^n \mathrm{P}(X_k \neq Y_{n,k}) \to 0$ as $n \to \infty$.

(c) Use (a) and (b) to conclude $\overline{X}_n \xrightarrow{\mathrm{P}} 0$.

3. Chebyshev's Law of Large Numbers. Let $X_1, X_2, \ldots$ be random variables with $\mathrm{E}(X_i) = \mu_i$, $\mathrm{Var}(X_i) = \sigma_i^2$ and $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$. Show that $\overline{X}_n - \bar{\mu}_n \xrightarrow{\mathrm{P}} 0$, provided

$$\frac{1}{n^2} \sum_{i=1}^n \sigma_i^2 \to 0 \quad \text{as } n \to \infty.$$

Compare to Bernstein's Law of Large Numbers, found in Exercise 4.3 in the text.