GAME THEORY

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INTRODUCTION.

Game theory is a fascinating subject. We all know many entertaining games, such as chess, poker, tic-tac-toe, bridge, baseball, computer games — the list is quite varied and almost endless. In addition, there is a vast area of economic games, discussed in Myerson (1991) and Kreps (1990), and the related political games, Ordeshook (1986), Shubik (1982), and Taylor (1995). The competition between firms, the conflict between management and labor, the fight to get bills through congress, the power of the judiciary, war and peace negotiations between countries, and so on, all provide examples of games in action. There are also psychological games played on a personal level, where the weapons are words, and the payoffs are good or bad feelings, Berne (1964). There are biological games, the competition between species, where natural selection can be modeled as a game played between genes, Smith (1982). There is a connection between game theory and the mathematical areas of logic and computer science. One may view theoretical statistics as a two person game in which nature takes the role of one of the players, as in Blackwell and Girshick (1954) and Ferguson (1968).

Games are characterized by a number of players or decision makers who interact, possibly threaten each other and form coalitions, take actions under uncertain conditions, and finally receive some benefit or reward or possibly some punishment or monetary loss. In this text, we present various mathematical models of games and study the phenomena that arise. In some cases, we will be able to suggest what courses of action should be taken by the players. In others, we hope simply to be able to understand what is happening in order to make better predictions about the future.

As we outline the contents of this text, we introduce some of the key words and terminology used in game theory. First there is the **number of players** which will be denoted by n. Let us label the players with the integers 1 to n, and denote the **set of players** by $N = \{1, 2, \ldots, n\}$. We study mostly two person games, $n = 2$, where the concepts are clearer and the conclusions are more definite. When specialized to one-player, the theory is simply called decision theory. Games of solitaire and puzzles are examples of one-person games as are various sequential optimization problems found in operations research, and optimization, (see Papadimitriou and Steiglitz (1982) for example), or linear programming, (see Chvátal (1983)), or gambling (see Dubins and Savage(1965)). There are even things called "zero-person games", such as the "game of life" of Conway (see

Berlekamp et al. (1982) Chap. 25); once an automaton gets set in motion, it keeps going without any person making decisions. We assume throughout that there are at least two players, that is, $n \geq 2$. In macroeconomic models, the number of players can be very large, ranging into the millions. In such models it is often preferable to assume that there are an infinite number of players. In fact it has been found useful in many situations to assume there are a continuum of players, with each player having an infinitesimal influence on the outcome as in Aumann and Shapley (1974). (Incidentally, both authors were later to win Nobel Prizes in Economics.) In this course, we take n to be finite.

There are three main mathematical models or forms used in the study of games, the **extensive form**, the **strategic form** and the **coalitional form**. These differ in the amount of detail on the play of the game built into the model. The most detail is given in the extensive form, where the structure closely follows the actual rules of the game. In the extensive form of a game, we are able to speak of a **position** in the game, and of a **move** of the game as moving from one position to another. The set of possible moves from a position may depend on the player whose turn it is to move from that position. In the extensive form of a game, some of the moves may be **random moves**, such as the dealing of cards or the rolling of dice. The rules of the game specify the probabilities of the outcomes of the random moves. One may also speak of the **information** players have when they move. Do they know all past moves in the game by the other players? Do they know the outcomes of the random moves?

When the players know all past moves by all the players and the outcomes of all past random moves, the game is said to be of **perfect information**. Two-person games of perfect information with win or lose outcome and no chance moves are known as **combinatorial games**. There is a beautiful and deep mathematical theory of such games. You may find an exposition of it in Conway (1976) and in Berlekamp et al. (1982). Such a game is said to be **impartial** if the two players have the same set of legal moves from each position, and it is said to be **partizan** otherwise. Part I of this text contains an introduction to the theory of impartial combinatorial games. For another elementary treatment of impartial games see the book by Guy (1989).

We begin Part II by describing the **strategic form** or normal form of a game. In the strategic form, many of the details of the game such as position and move are lost; the main concepts are those of a strategy and a payoff. In the strategic form, each player chooses a strategy from a set of possible strategies. We denote the **strategy set** or **action space** of player i by A_i, for $i = 1, 2, \ldots, n$. Each player considers all the other players and their possible strategies, and then chooses a specific strategy from his strategy set. All players make such a choice simultaneously, the choices are revealed and the game ends with each player receiving some payoff. Each player's choice may influence the final outcome for all the players.

We model the payoffs as taking on numerical values. In general the payoffs may be quite complex entities, such as "you receive a ticket to a baseball game tomorrow when there is a good chance of rain, and your raincoat is torn". The mathematical and philosophical justification behind the assumption that each player can replace such payoffs with numerical values is discussed in the Appendix under the title, **Utility Theory**. This theory is treated in detail in the books of Savage (1954) and of Fishburn (1988). We therefore assume that each player receives a numerical payoff that depends on the actions chosen by all the players. Suppose player 1 chooses $a_1 \in A_1$, player 2 chooses $a_2 \in A_2$, etc. and player n chooses $a_n \in A_n$. Then we denote the payoff to player j, for $j = 1, 2, \ldots, n$, by $f_j(a_1, a_2, \ldots, a_n)$, and call it the **payoff function for player** j.

The **strategic form of a game** is defined then by the three objects:

(1) the set, $N = \{1, 2, \ldots, n\}$, of players,

(2) the sequence, $A_1, \ldots, A_n$, of strategy sets of the players, and

(3) the sequence, $f_1(a_1, \ldots, a_n), \ldots, f_n(a_1, \ldots, a_n)$, of real-valued payoff functions of the players.

A game in strategic form is said to be **zero-sum** if the sum of the payoffs to the players is zero no matter what actions are chosen by the players. That is, the game is zero-sum if

$$\sum_{i=1}^{n} f_i(a_1, a_2, \ldots, a_n) = 0$$

for all $a_1 \in A_1$, $a_2 \in A_2, \ldots$, $a_n \in A_n$. In the first four chapters of Part II, we restrict attention to the strategic form of finite, two-person, zero-sum games. Such a game is said to be **finite** if both the strategy sets are finite sets. Theoretically, such games have clear-cut solutions, thanks to a fundamental mathematical result known as the **minimax theorem**. Each such game has a **value**, and both players have **optimal strategies** that guarantee the value.

In the last three chapters of Part II, we treat two-person zero-sum games in extensive form, and show the connection between the strategic and extensive forms of games. In particular, one of the methods of solving extensive form games is to solve the equivalent strategic form. Here, we give an introduction to Recursive Games and Stochastic Games, an area of intense contemporary development (see Filar and Vrieze (1997), Maitra and Sudderth (1996) and Sorin (2002)). In the last chapter, we investigate the problems that arise when at least one of the strategy sets of the players is an infinite set.

In Part III, the theory is extended to two-person **non-zero-sum** games. Here the situation is more nebulous. In general, such games do not have values and players do not have optimal strategies. The theory breaks naturally into two parts. There is the **noncooperative theory** in which the players, if they may communicate, may not form binding agreements. This is the area of most interest to economists, see Gibbons (1992), and Bierman and Fernandez (1993), for example. In 1994, John Nash, John Harsanyi and Reinhard Selten received the Nobel Prize in Economics for work in this area. Such a theory is natural in negotiations between nations when there is no overseeing body to enforce agreements, and in business dealings where companies are forbidden to enter into agreements by laws concerning constraint of trade. The main concept, replacing value and optimal strategy is the notion of a **strategic equilibrium**, also called a **Nash equilibrium**. This theory is treated in the first three chapters of Part III.

On the other hand, in the **cooperative theory** the players are allowed to form binding agreements, and so there is strong incentive to work together to receive the largest total payoff. The problem then is how to split the total payoff between or among the players. This theory also splits into two parts. If the players measure utility of the payoff in the same units and there is a means of exchange of utility such as **side payments**, we say the game has **transferable utility**; otherwise **non-transferable utility**. The last chapter of Part III treat these topics.

When the number of players grows large, even the strategic form of a game, though less detailed than the extensive form, becomes too complex for analysis. In the **coalitional form** of a game, the notion of a strategy disappears; the main features are those of a **coalition** and the **value** or **worth** of the coalition. In many-player games, there is a tendency for the players to form coalitions to favor common interests. It is assumed each coalition can guarantee its members a certain amount, called the value of the coalition. The coalitional form of a game is a part of cooperative game theory with transferable utility, so it is natural to assume that the **grand coalition**, consisting of all the players, will form, and it is a question of how the payoff received by the grand coalition should be shared among the players. We will treat the coalitional form of games in Part IV. There we introduce the important concepts of the **core** of an economy. The core is a set of payoffs to the players where each coalition receives at least its value. An important example is two-sided matching treated in Roth and Sotomayor (1990). We will also look for principles that lead to a unique way to split the payoff from the grand coalition, such as the **Shapley value** and the **nucleolus**. This will allow us to speak of the power of various members of legislatures. We will also examine cost allocation problems (how should the cost of a project be shared by persons who benefit unequally from it).

Related Texts. There are many texts at the undergraduate level that treat various aspects of game theory. Accessible texts that cover certain of the topics treated in this text are the books of Straffin (1993), Morris (1994) and Tijs (2003). The book of Owen (1982) is another undergraduate text, at a slightly more advanced mathematical level. The economics perspective is presented in the entertaining book of Binmore (1992). The New Palmgrave book on game theory, Eatwell et al. (1987), contains a collection of historical sketches, essays and expositions on a wide variety of topics. Older texts by Luce and Raiffa (1957) and Karlin (1959) were of such high quality and success that they have been reprinted in inexpensive Dover Publications editions. The elementary and enjoyable book by Williams (1966) treats the two-person zero-sum part of the theory. Also recommended are the lectures on game theory by Robert Aumann (1989), one of the leading scholars of the field. And last, but actually first, there is the book by von Neumann and Morgenstern (1944), that started the whole field of game theory.

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