The Symmetric Exclusion Process:
Correlation Inequalities and Applications

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**Pemantle’s problem** (2000): Let $\eta_t$ be a symmetric exclusion process on $\mathbb{Z}^1$ with transition probabilities

$$p(x, y) = p(y, x) = p(y - x)$$

and $\eta_0 : \cdots 1 1 1 0 0 0 \cdots$, and let

$$N_t = \sum_{x>0} \eta_t(x).$$

Is it true that

$$\frac{N_t - EN_t}{[\text{var}(N_t)]^{1/2}} \Rightarrow N(0, 1)?$$
Negative Correlations

Andjel (1988): If $A \cap B = \emptyset$, then

$$P^n(\eta_t \equiv 1 \text{ on } A, \eta_t \equiv 1 \text{ on } B) \leq P^n(\eta_t \equiv 1 \text{ on } A)P^n(\eta_t \equiv 1 \text{ on } B).$$

The same proof does not give

$$P^n(\eta_t \equiv 1 \text{ on } A, \eta_t \equiv 0 \text{ on } B) \geq P^n(\eta_t \equiv 1 \text{ on } A)P^n(\eta_t \equiv 0 \text{ on } B).$$

Now we know:

**Theorem.** For any symmetric exclusion process with deterministic (or product) initial distribution,

(a) $\eta_t$ is negatively associated (NA), i.e.,

$$Ef(\eta_t)g(\eta_t) \leq Ef(\eta_t)Eg(\eta_t)$$

for all $f, g \uparrow$ depending on disjoint sets of coordinates, and

(b) $\forall T$, $\exists$ independent Bernoulli random variables $\zeta(x)$ so that

$$\sum_{x \in T} \eta_t(x) \quad \text{and} \quad \sum_{x \in T} \zeta(x)$$

have the same distribution.

**Corollary.** If $\text{var}(N_t) \to \infty$, then $N_t$ satisfies the CLT.
The Strong Rayleigh Property

The generating polynomial of a p.m. $\mu$ on $\{0, 1\}^n$ is

$$f(z_1, \ldots, z_n) = E_\mu \prod_{i=1}^{n} z_i^{\eta(i)}.$$ 

Then

$$\frac{\partial f}{\partial z_i} \bigg|_{z_k \equiv 1} = E_\mu \eta(i), \quad \frac{\partial^2 f}{\partial z_i \partial z_j} \bigg|_{z_k \equiv 1} = E_\mu \eta(i) \eta(j).$$

Pairwise negative correlations is equivalent to

(*) $$f(z) \frac{\partial^2 f}{\partial z_i \partial z_j}(z) \leq \frac{\partial f}{\partial z_i}(z) \frac{\partial f}{\partial z_j}(z)$$

for $z_k \equiv 1$.

**Definitions.** (a) $\mu$ is **Strong Rayleigh** (SR) if (*) holds for all $z \in R^n$.

(b) $\mu$ is **stable** if $f \neq 0$ if $\Im(z_k) > 0$ for all $k$.

**Remark.** If $\mu = \nu_\alpha$ is a product measure, then

$$f(z) = \prod_{i=1}^{n} [\alpha_i z_i + (1 - \alpha_i)],$$

so $\mu$ is SR — (*) holds with equality — and stable.
Results about the Strong Rayleigh Property

Theorem. (Bräändén (2007)) SR is equivalent to stability.

Why is this true? Think of it as an analogue of the quadratic formula for the roots of $ax^2 + bx + c$: Stability is a statement about whether there are roots in the upper half plane. SR is like a discriminant condition.

Theorem. $SR \implies NA$.

Proof. Based on the Feder-Mihail (1992) proof of NA for the uniform spanning tree measure. Easier if $\sum_x \eta(x)$ is constant.

Key use of SR property: If $\mu$ is SR on $\{0,1\}^n$, then so is its “symmetric homogenization” on $\{0,1\}^{2n}$, which satisfies $\sum_x \eta(x)$ constant.

Theorem. If the initial distribution of a symmetric exclusion process is SR, then so is the distribution at time $t$.

Proof. It is sufficient to prove it for exclusion on two sites, i.e. that stability is preserved by the transformation:

$$\mu \rightarrow T\mu = p\mu + (1-p)\mu_{i,j}.$$  

($\mu_{i,j}$ is obtained from $\mu$ by permuting $\eta(i), \eta(j)$.)
Suppose $f$ is stable. Need to show that if $\Im(z_k) > 0$ for all $k$, then $Tf(z) \neq 0$. Fix $z_k$ for $k \neq i, j$. Need to show that $T$ preserves stability of polynomials of the form $h(z, w) = a + bz + cw + dzw$. with complex $a, b, c, d$. If not all coefficients are zero, $h$ is stable iff

$$\Re(b\bar{c} - a\bar{d}) \geq |bc - ad|,$$
$$\Im(ab) \geq 0, \Im(ac) \geq 0, \Im(bd) \geq 0, \Im(cd) \geq 0.$$

**Theorem.** If the distribution of $\{\eta(i), 1 \leq i \leq n\}$ is SR, then there exist independent Bernoulli $\{\zeta(i), 1 \leq i \leq n\}$ so that

$$\sum_i \eta(i) \quad \text{and} \quad \sum_i \zeta(i)$$

have the same distribution.

**Proof.** $f(z, z, ..., z) = Ez\sum \eta(i)$ is not zero if $Im(z) > 0$ or if $Im(z) < 0$ or if $z > 0$. So all roots are negative:

$$Ez\sum \eta(i) = \prod_i [\alpha_i z + (1 - \alpha_i)],$$

where the roots are $-(1 - \alpha_i)/\alpha_i$. 
Back to the CLT for the Exclusion Process

Theorem. Suppose $\sigma^2 = \sum_n n^2 p(n) < \infty$. Then

$$\frac{N_t - EN_t}{\sqrt{\text{var}(N_t)}} \Rightarrow N(0, 1).$$

Furthermore,

$$\lim_{t \to \infty} \frac{EN_t}{\sqrt{t}} = \frac{\sigma}{\sqrt{2\pi}} \quad \text{and} \quad 0 < c_1 \leq \frac{\text{var}(N_t)}{\sqrt{t}} \leq c_2 < \infty.$$

Proof. Need to consider the first two moments of $N_t$. Let $X_t, Y_t$ be independent copies of the random walk. By duality,

$$EN_t = \sum_{x>0} P(\eta_t(x) = 1) = \sum_{x>0} P^x(X_t \leq 0) = E^0 X_t^+.$$

Similarly,

$$\sum_{x>0} [P(\eta_t(x) = 1)]^2 = E^{(0,0)} \min(X_t^+, Y_t^+).$$

So,

$$\sum_{x>0} \text{var}(\eta_t(x)) \sim \frac{\sigma}{2\sqrt{\pi}} \sqrt{t}.$$
Let
\[ K(t) = - \sum_{x, y > 0, x \neq y} \text{cov}(\eta_t(x)\eta_t(y)). \]

Then if \( f(x, y) = 1_{\{x, y \leq 0\}} \),
\[ K(t) = \sum_{x, y > 0; x \neq y} [U(t) - V(t)]f(x, y) \]
\[ = \sum_{x, y > 0; x \neq y} \int_0^t V(t - s)(U - V)U(s)f(x, y)ds \]
\[ \leq \int_0^t \sum_{x < y} p(x, y) [P^0(x \leq X_s < y)]^2 \gamma(t - s, x, y)ds, \]

where
\[ \gamma(t, x, y) = P^0(X_t < x)P^0(X_t < y) + P^0(X_t \geq x)P^0(X_t \geq y). \]

Using \( \gamma(t, x, y) \leq 1 \) leads to
\[ \limsup_{t \to \infty} \frac{K(t)}{\sqrt{t}} \leq \frac{\sigma}{2\sqrt{\pi}}. \]

Being more careful, one gets
\[ \limsup_{t \to \infty} \frac{K(t)}{\sqrt{t}} < \frac{\sigma}{2\sqrt{\pi}}. \]
Poisson Convergence

**Theorem.** Suppose the Bernoulli random variables \( \{\eta_n(x)\} \) are strong Rayleigh for each \( n \). If

\[
\lim_{n \to \infty} \sum_x E\eta_n(x) = \lambda, \quad \lim_{n \to \infty} \sum_x [E\eta_n(x)]^2 = 0,
\]

and

\[
\lim_{n \to \infty} \sum_{x \neq y} Cov(\eta_n(x), \eta_n(y)) = 0,
\]

then

\[
\sum_x \eta_n(x) \Rightarrow \text{Poisson}(\lambda).
\]

Application to Symmetric Exclusion

Recall that the extremal invariant measures \( \mu_\alpha \) are in one to one correspondence with harmonic functions \( \alpha(x) \) for \( P \) with \( 0 \leq \alpha(x) \leq 1 \), and

\[
\mu_\alpha = \lim_{t \to \infty} \nu_\alpha S(t).
\]

Furthermore, \( \mu_\alpha = \nu_\alpha \) iff \( \alpha \) is constant.

**Theorem.** \( \mu_\alpha \) is SR, and hence NA
Example

Let \( P \) be simple random walk on the binary tree:

\[
l(x) : 2 \quad 1 \quad 0 \quad 0 \quad 1 \quad 2
\]

**Theorem.** Suppose

\[
\alpha(x) = \begin{cases} 
\frac{1}{3 \cdot 2^l(x)} & \text{if } x \in L, \\
1 - \frac{1}{3 \cdot 2^l(x)} & \text{if } x \in R.
\end{cases}
\]

Then with respect to \( \mu_{\alpha} \),

\[
\sum_{x \in L : l(x) = n} \eta(k) \Rightarrow \text{Poisson} \left( \frac{1}{3} \right)
\]

\[
\sigma_n^{-1} \left[ \sum_{x \in L : l(k) < n} \eta(x) - \frac{n}{3} \right] \Rightarrow N(0, 1),
\]

where \( \frac{23}{189} \leq \sigma_n^2/n \leq \frac{1}{3} \) asymptotically.