TAGGED PARTICLE DISTRIBUTIONS
OR
HOW TO CHOOSE A HEAD AT RANDOM

Thomas M Liggett

University of California, Los Angeles

November 16, 2000

Abstract. Thorisson and others have proved results that imply the following: given an i.i.d. family of Bernoulli random variables indexed by Z^d, there exists an occupied site $X \in Z^d$ with the property that relative to its location, the other variables are still i.i.d. We raise the question of how large such an X must be. For $d = 1$, we prove that any X with this property satisfies $E|X|^{\frac{1}{2}} = \infty$. Moreover, there does exist such an X with tails $P(|X| \geq n)$ of order $Cn^{-\frac{1}{2}}$, so these results are essentially best possible. Analogous results for the Poisson process in one dimension are given. The corresponding problem for stationary ergodic sequences is considered also. This project was motivated by some tagged particle problems.

1. Introduction

Take $0 < \rho < 1$, and let ν_ρ be the product measure on $\{0,1\}^{Z^d}$ with density ρ. Consider the following problems:

Problem A: Given $\eta \in \{0,1\}^{Z^d}$ with distribution ν_ρ, find a Z^d–valued random variable X (possibly using additional randomization) so that $\eta(X) = 1$ a.s. and $\{\eta(X+k), k \neq 0\}$ are i.i.d. with density ρ.

Problem B: What is the optimal tail behavior of a solution X to Problem A?

Now let $\{\Pi(A), A \text{ Borel } \subset R^d\}$ be a rate one spatial Poisson process. The following are continuous versions of Problems A and B:

Problem C: Find $X \in R^d$ so that $\Pi(\{X\}) = 1$ and $\{\Pi(X+A), A \text{ Borel } \subset R^d\}$ is a rate one spatial Poisson process with an added point at 0.

Problem D: What is the optimal tail behavior of a solution X to Problem C?

1991 *Mathematics Subject Classification.* 60G60, 60K35, 60B99.

Key words and phrases. Product measure, tagged particles, exclusion process, Poisson process, shift coupling, palm measure.

Research supported in part by NSF Grants DMS 9703830 and DMS 0070465.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

These problems are motivated by tagged particle motions in the context of exclusion processes, and arose in conversations among Stefano Olla, Pablo Ferrari and Sunder Sethuraman. For their purposes, they needed to find a solution X to Problem A with a finite first moment – an impossible objective in one dimension, as we will see. Their original problem involved the asymptotic variance for the position of a tagged paricle in a zero range process, but their discussions led to the exclusion context. As is well known, there is a close connection between exclusion and zero range processes in one dimension – see Kipnis (1986) for an early reference.

A connection between Problem A and tagged particle motions is the following. Let η_t be the exclusion process on Z^d with transition probabilities $p(j,k) = p(0, k-j)$. Start the process with the distribution in which $\eta(0) = 1$ a.s., and $\{\eta(k), k \neq 0\}$ are i.i.d. Bernoulli with parameter ρ. Let X_t be the position at time t of the particle that began at 0. (This is the "tagged particle".) Then the process ζ_t defined by $\zeta_t(k) = \eta_t(X_t + k)$ is stationary. (See Proposition 4.3 of Part III of Liggett (1999), for example.) On the other hand, ν_ρ is an invariant measure for η_t (Theorem 1.2 of Part III of Liggett (1999)). Therefore, Problem A is essentially asking whether it is possible to start the process in such a way that both η_t and ζ_t are in equilibrium.

Our problems are closely related to results in a number of recent papers on what is known as "shift-coupling": Aldous and Thorisson (1993), Thorisson (1995), Thorisson (1996), Georgii (1997), Derrida, Goldstein, Lebowitz and Speer (1998), and Thorisson (1999). In fact, the existence of solutions to Problems A and C is a consequence of much more general results in those papers – see Theorem B.1 of Thorisson (1995) for Problem A in one dimension, for example, and Thorisson (1996) for Problems A and C in general dimensions. None of these papers discusses how big the solution X must be. Our interest in this paper is in determining whether there are solutions to Problems A and C for which X has tails that are not too large – i.e., Problems B and D.

Before describing our results we note that it is not possible to find an X so that

$$\{\eta(X+k), k \neq 0\}$$

are i.i.d. with density ρ *conditionally* on X. To see this, assume that such a construction is possible, and write

$$\begin{aligned} P\big(\eta(k)=1\big) =& \sum_j P\big(\eta(k)=1 \mid X=j\big)P\big(X=j\big) \\ =&\rho\sum_{j\neq k} P\big(X=j\big) + P\big(X=k\big) \\ =&\rho + (1-\rho)P\big(X=k\big). \end{aligned}$$

For this to be equal to ρ for every k, we would need $P\big(X=k\big) = 0$ for every k, which is impossible.

It is perhaps also worth mentioning at the outset that Problems A and C are more subtle than they might appear to be at first glance. Take Problem C, for example. The naive first guess is that $X =$ the first point in Π to the right of the origin, might be a solution. If it were, it would have exponential tails. But of course the difficulty is that the distance from X to the first point in Π to the left of X has a distribution that is that of the sum of two unit exponentials, when it should be just one exponential. One might try the following variant of this failed solution: Let Z be any positive integer valued random variable that is independent of Π, and let X be the Zth point in Π to the right of the origin. Perhaps one can choose the distribution of Z so that X has the required properties. In fact, no such choice works. To see this, let $J_1, J_2, \ldots$ be the lengths of the intervals between successive points of Π to the left of X. Then conditionally on $Z = k$, the J_i's are independent, with

$$P(J_i > t) = \begin{cases} e^{-t} & \text{if } i \neq k, \\ (1+t)e^{-t} & \text{if } i = k. \end{cases}$$

Therefore

$$P\big(J_i > t\big) = e^{-t} + te^{-t}P\big(Z = i\big).$$

In Section 2 we prove results that imply that solutions to Problem A for some values of ρ lead to solutions to it for other values of ρ. In Section 3, we prove a negative result, which implies in particular that the Olla-Ferrari-Sethuraman problem (with finite first moment requirement) has no solution in one dimension. Section 4 is devoted to explicit constructions of solutions to Problems A and C in one dimension, and the determination of the tail behavior of these solutions. These constructions are motivated by the necessary condition in Theorem 3.1. Together, these results provide sharp answers to Problems B and D in one dimension:

Theorem 1.1. *Take $d = 1$.*
(a) Any solution X to Problem A satisfies

$$\liminf_{t\to\infty} \frac{E\big(|\rho X| \wedge t\big)}{\sqrt{t}} \geq \sqrt{\frac{1-\rho}{2\pi}},$$

while any solution X to Problem C satisfies

$$\liminf_{t\to\infty} \frac{E\big(|X| \wedge t\big)}{\sqrt{t}} \geq \sqrt{\frac{1}{2\pi}}.$$

In particular, in both cases, X satisfies $E|X|^{\frac{1}{2}} = \infty$.
(b) There exists a solution X to Problem A that satisfies

$$\lim_{t\to\infty} \frac{E\big(|N^{-1}X| \wedge t\big)}{\sqrt{t}} \leq 8\sqrt{\frac{1-N^{-1}}{2\pi}},$$

where the integer N is determined by $\frac{1}{N} \leq \rho < \frac{1}{N-1}$. There is a solution X to Problem C that satisfies

$$\lim_{t\to\infty} \frac{E\big(|X| \wedge t\big)}{\sqrt{t}} \leq 8\sqrt{\frac{1}{2\pi}}.$$

Here is a construction that has the properties stated above in the case of the Poisson process. Let

$$\sigma = \min\{k \geq 1 : \Pi(0,k) \geq k\},$$

and if $\sigma = k$, take X to be the kth point in Π to the right of the origin. For Problem A with $\rho = \frac{1}{N}$, the construction is similar, except that now

$$\sigma = \min\{k \geq 1 : \sum_{j=0}^{kN-1} \eta(j) \geq k\}. \tag{1.2}$$

When $\rho = \frac{1}{2}$, the construction can be described somewhat more simply: Taking σ as in (1.2) with $N = 2$, ($2\sigma - 1$ is essentially an excursion for the symmetric simple random walk $2\sum_{j=0}^{n-1} \eta(j) - n$), note that $\eta(0) = 0$ implies that $\eta(2\sigma - 1) = 1$. Then

$$X = \begin{cases} 0 & \text{if } \eta(0) = 1 \\ 2\sigma - 1 & \text{if } \eta(0) = 0. \end{cases} \tag{1.3}$$

In Section 5, we discuss versions of Problems A and B for measures other than product measures. Perhaps surprisingly, the construction described in (1.2) and (1.3) works in great generality. In particular, if μ is the distribution of a stationary renewal process with spacing distribution in the domain of normal attraction of a (completely asymmetric) stable law of index $\alpha \in (1, 2]$, then an analogue of Theorem 1.1 holds with $\sqrt{t}$ replaced by $t^{\frac{1}{\alpha}}$ (and different constants).

In an appendix, we determine the tail behavior of a construction based on the method of common components which has been used in virtually all previous treatments of Problems A and C.

The main results in this paper deal only with the one dimensional problem. It is natural to ask for an analogue of Theorem 1.1 in higher dimensions. A reasonable approach to making a construction for $d > 1$ would appear to be the following: Let

$$f : Z^1 \to Z^d$$

be one-to-one. It is easy to choose the f so that

$$|f(x)|^d \leq C|x|, \tag{1.4}$$

though we will not assume that yet. Given $\eta \in \{0, 1\}^{Z^d}$ with distribution $\nu_{\frac{1}{2}}$, for example, define $\zeta \in \{0, 1\}^{Z^1}$ with distribution $\nu_{\frac{1}{2}}$ by $\zeta(x) = \eta\big(f(x)\big)$. Then apply the one dimensional construction to ζ, calling the result Y, and let $X = f(Y)$. One might guess that X solves Problem A for η, but this is not necessarily the case. For example, take $d = 2$ and

$$f(n) = \begin{cases} (n, 0) & \text{if } n \leq 0 \\ (n-1, 1) & \text{if } n > 0. \end{cases}$$

Then

$$P\big(\eta(X - (1, 0)) = 0\big) =$$
$$P\big(\eta(0, 0) = 1, \eta(-1, 0) = 0\big) + P\big(\eta(0, 0) = 0, \eta(0, 1) = 1, \eta(-1, 1) = 0\big) = \frac{3}{8},$$

when it should be $\frac{1}{2}$. If the above approach had worked, such a construction with an f satisfying (1.4) would have lead to a solution to Problem A satifying

$$P\big(|X| \geq t\big) \leq Ct^{-\frac{d}{2}}.$$

Since this simpleminded approach does not work, a new technique is needed in higher dimensions. This is the subject of a forthcoming paper with Ander Holroyd.

2. Comparisons

This section is devoted to showing how solutions to Problem A for one value of ρ can be used to construct solutions for other values of ρ. This is necessary, since our constructions in Section 4 apply directly only for certain values of ρ. In this section, the dimension d is general.

Proposition 2.1. *Suppose X is a solution to Problem A for ν_ρ, and let $\lambda > \rho$. Then there is a solution Y to Problem A for ν_λ with Y having the same distribution as X.*

Proof. Let η, ζ be independent, with distributions ν_ρ and $\nu_{\rho'}$ respectively, where

$$\rho' = \frac{\lambda - \rho}{1 - \rho}.$$

Then the coordinatewise maximum $\eta \vee \zeta$ has distribution ν_λ, since $(1-\rho)(1-\rho') = 1 - \lambda$. With this construction, any solution X to Problem A for η (chosen so as to be independent of ζ) is automatically a solution to Problem A for ν_λ. To see this, let K be a finite subset of $Z^d \backslash \{0\}$ and write

$$\begin{aligned}
P\big((\eta \vee \zeta)(X+k) = 0 \ \forall\ k \in K\big) =& \sum_j P\big((\eta \vee \zeta)(j+k) = 0 \ \forall\ k \in K, X = j\big) \\
=& \sum_j P\big(\eta(j+k) = 0, \zeta(j+k) = 0 \ \forall\ k \in K, X = j\big) \\
=& (1-\rho')^{|K|} \sum_j P\big(\eta(j+k) = 0 \ \forall\ k \in K, X = j\big) \\
=& (1-\rho')^{|K|} P\big(\eta(X+k) = 0 \ \forall\ k \in K\big) \\
=& (1-\rho')^{|K|}(1-\rho)^{|K|} = (1-\lambda)^{|K|}.
\end{aligned}$$

Here $|K|$ denotes the cardinality of K. Of course, $(\eta \vee \zeta)(X) \geq \eta(X) = 1$. ■

Next, it is natural to try to go from solutions for large ρ to solutions for small ρ. In order to do so, we need to introduce some notation and terminology. If $j, k \in Z^d$ we will write $j < k$ to mean that all the coordinates satisfy the corresponding inequality: $j_i < k_i$ for all $1 \leq i \leq d$. Given η with distribution ν_ρ say that a solution X to Problem A is *optional* if $X > 0$ a.s. and X is independent of the collection $\{\eta(X+k), k > 0\} \cup \{\eta(k), k \not> 0\}$.

Proposition 2.2. *Suppose that $\lambda < \rho$ and that there is an optional solution X to Problem A for ζ whenever ζ has distribution ν_ρ. Then there is a solution Y to Problem A for ν_λ, where Y has the distribution of $X_0 + X_1 + \cdots + X_T$, X_i are i.i.d. with the distribution of X, T is independent of the X_i's, and*

$$P(T \geq n) = \left(\frac{\rho - \lambda}{\rho(1-\lambda)}\right)^n, \quad n \geq 0.$$

Proof. Let $\eta_0, \zeta_0, \zeta_1, \zeta_2, ...$ be independent with distributions ν_λ for η_0 and $\nu_{\rho'}$ for the ζ_i's, where

$$\rho' = \frac{\rho - \lambda}{1 - \lambda}.$$

Then $(\eta_0 \vee \zeta_0)$ has distribution ν_ρ. Let X_0 be an optional solution to Problem A for $(\eta_0 \vee \zeta_0)$. Then $\{\eta_0(X_0+k), k \in Z^d\}$ are independent (since the coordinates of η_0 are conditionally independent given $(\eta_0 \vee \zeta_0)$), $\{\eta_0(X_0), X_0\}$ are independent (since $(\eta_0 \vee \zeta_0)(X_0) = 1$), and

$$P\big(\eta_0(X_0+k)=1\big) = \begin{cases} \lambda & \text{if } k \neq 0, \\ \lambda\rho^{-1} & \text{if } k = 0. \end{cases}$$

The latter value comes from the fact that

$$P\big(\eta_0(j)=1 \mid (\eta_0 \vee \zeta_0)(j)=1\big) = \frac{\lambda}{\lambda+\rho'-\lambda\rho'} = \frac{\lambda}{\rho}.$$

Since $\lambda\rho^{-1} > \lambda$, we can take Bernoulli random variables σ_0, τ_0 independent of each other and of $\big\{\eta_0(X_0+k), k \in Z^d\backslash\{0\}\big\}$ so that $P(\sigma_0 = 1) = \lambda$ and $\eta_0(X_0) = \sigma_0 \vee \tau_0$. Then define η_1 with distribution ν_λ by

$$\eta_1(k) = \begin{cases} \eta_0(X_0+k) & \text{if } k \neq 0, \\ \sigma_0 & \text{if } k = 0. \end{cases}$$

Note that η_1 is independent of τ_0 and that

$$P(\tau_0 = 0) = \frac{\rho-\lambda}{\rho(1-\lambda)}. \tag{2.3}$$

Also, since $\{\eta_0(X_0), X_0\}$ are independent, we have that $\{\tau_0, X_0\}$ are independent.

Now let X_1 be an optional solution (using the same construction that produced X_0) to Problem A for $\eta_1 \vee \zeta_1$, and continue in this way. Note that since the solutions are optional at each stage, the resulting $X_0, X_1, ...$ are i.i.d. Furthermore, at each stage, $\{\eta_n(X_n+k), k \in Z^d\}$ are independent, are independent of $\tau_0, ..., \tau_{n-1}$, and

$$P\big(\eta_n(X_n+k)=1\big) = \begin{cases} \lambda & \text{if } k \neq 0, \\ \lambda\rho^{-1} & \text{if } k = 0. \end{cases}$$

Since τ_n is defined in terms of $\eta_n(X_n)$, $\tau_0, \tau_1, ...$ are i.i.d., and are independent of $\{X_0, X_1, ...\}$. Let

$$S_n = X_0 + \cdots + X_{n-1}.$$

Then we can solve recursively to get

$$\eta_n(k) = \begin{cases} \eta_0(S_n+k) & \text{if } k+S_n \notin \{S_1, ..., S_n\}, \\ \sigma_{i-1} & \text{if } k+S_n = S_i. \end{cases} \tag{2.4}$$

Here $i = 1, ..., n$. (Note that the S_i's are distinct since $X > 0$.) In particular, $\eta_n(X_n) = \eta_0(S_{n+1})$.

Take $T = \min\{n \geq 0 : \tau_n = 1\}$. The tails of the distribution of T are correct by (2.3) and the fact that the τ_n's are i.i.d. It remains to verify that $Y = S_{T+1}$ is a solution to Problem A for η_0. Since $\tau_T = 1$ and

$$\tau_n \vee \sigma_n = \eta_n(X_n) = \eta_0(S_{n+1}),$$

$\eta_0(Y) = 1$ a.s.

Finally, let $K \subset Z^d \backslash \{0\}$ be finite, and write for $n \geq 1$

$$P\big(\eta_0(Y+k) = 1 \ \forall \ k \in K, T = n-1\big) = P\big(\eta_0(S_n+k) = 1 \ \forall \ k \in K, T = n-1\big). \tag{2.5}$$

To compute the right side of (2.5), note first that

$$T > m \quad \Rightarrow \quad \tau_m = 0 \quad \Rightarrow \quad \eta_0(S_{m+1}) = \eta_m(X_m) = \sigma_m \vee \tau_m = \sigma_m. \tag{2.6}$$

Therefore, on the event $\{T = n-1\}$,

$$k + S_n \notin \{S_1, ..., S_n\} \quad \Rightarrow \quad \eta_0(S_n + k) = \eta_n(k)$$

by (2.4), and for $0 \leq i < n$,

$$k + S_n = S_i \quad \Rightarrow \quad \eta_0(S_n+k) = \eta_0(S_i) = \sigma_{i-1} = \eta_n(k) \tag{2.7}$$

by (2.6). Therefore (recalling that $0 \notin K$, so the case $i = n$ does not arise in (2.7)) the right side of (2.5) is just

$$P\big(\eta_n(k) = 1 \ \forall \ k \in K, T = n-1\big). \tag{2.8}$$

Since $\{T = n-1\}$ depends only on $\tau_0, ..., \tau_{n-1}$, it is independent of η_n. So, (2.8) becomes $\lambda^{|K|}P(T = n-1)$. Recalling that this is the same as the left side of (2.5), and summing on n, gives

$$P\big(\eta_0(Y+k) = 1 \ \forall \ k \in K\big) = \lambda^{|K|}$$

as required. ■

3. Necessary Conditions in One Dimension

This section is devoted to the proof of Theorem 1.1(a), which we restate below. We begin with some intuition that could be made into a proof of a somewhat weaker result. Suppose there were a solution to Problem A with finite first moment. Then this construction of X could be carried out repeatedly, leading to the conclusion that the density of 1's in η is greater than it should be according to the strong law of large numbers. The point is that any site that is chosen by one of the X's is occupied for sure, while the other sites are occupied with probability ρ. The finite mean assumption guarantees that a positive fraction of sites are occupied "for sure". An extension of this argument suggests that a solution X to Problem A that satisfies $E|X|^{\frac{1}{2}} < \infty$ would contradict the central limit theorem for η. Here is the result, with a different proof.

Theorem 3.1. *Suppose $d = 1$. Any solution X to Problem A satisfies*

$$\liminf_{t \to \infty} \frac{E\big(|\rho X| \wedge t\big)}{\sqrt{t}} \geq \sqrt{\frac{1-\rho}{2\pi}},$$

while any solution X to Problem C satisfies

$$\liminf_{t\to\infty} \frac{E\big(|X| \wedge t\big)}{\sqrt{t}} \geq \sqrt{\frac{1}{2\pi}}.$$

In particular, in both cases, X satisfies $E|X|^{\frac{1}{2}} = \infty$.

Proof. The proof is based on the shift-coupling inequality, which can be found in Chapter 7 of Thorisson (2000). To keep the proof self-contained, we will state and prove the version we need below. Consider first the case of Problem A. Define the shift T on $\{0,1\}^Z$ by $(T\eta)(j) = \eta(j+1)$. The shift acts on measures via $(T\mu)(A) = \mu(T^{-1}A)$. The total variation norm on measures is defined as usual:

$$||\mu|| = \sup\left\{ \int f d\mu : |f| \leq 1 \right\}.$$

Suppose then that μ_1 and μ_2 are probability measures on $\{0,1\}^{Z^1}$, and that random $\eta \in \{0,1\}^{Z^1}$ and $X \in Z^1$ are chosen so that η has distribution μ_1 and $T^X\eta$ has distribution μ_2. The shift-coupling inequality asserts that

$$\left|\left| \frac{1}{n}\sum_{k=1}^n T^k\mu_1 - \frac{1}{n}\sum_{k=1}^n T^k\mu_2 \right|\right| \leq \frac{2}{n} E\big[|X| \wedge n\big] \tag{3.2}$$

for $n \geq 1$. To check this, take f satisfying $|f| \leq 1$ and write

$$\begin{aligned}\left| \frac{1}{n}\sum_{k=1}^n Ef\big(T^k\eta\big) - \frac{1}{n}\sum_{k=1}^n Ef\big(T^{X+k}\eta\big) \right| &= \frac{1}{n}\left| E\sum_l f\big(T^l\eta\big)\Big[1_{[1,n]}(l) - 1_{[X+1,X+n]}(l)\Big] \right| \\ &\leq \frac{1}{n} E\sum_l \big|1_{[1,n]}(l) - 1_{[X+1,X+n]}(l)\big| \\ &= \frac{2}{n} E\big[|X| \wedge n\big].\end{aligned}$$

Taking the supremum over these f's gives (3.2).

Next, we will compute the left side of (3.2) in case $\mu_1 = \nu_\rho$ and μ_2 is the conditional measure $\nu_\rho(\,\cdot \mid \eta(0) = 1)$. Since μ_1 is shift invariant and $T^k\mu_2$ is absolutely continuous with respect to μ_1 for each k, the left side of (3.2) can be written as

$$\int \left| 1 - \frac{1}{n}\sum_{k=1}^n \frac{d(T^k\mu_2)}{d\mu_1} \right| d\mu_1. \tag{3.3}$$

Since $T^k\mu_2(\,\cdot\,) = \nu_\rho(\,\cdot \mid \eta(-k) = 1)$,

$$\frac{d(T^k\mu_2)}{d\mu_1}(\eta) = \rho^{-1}\eta(-k).$$

Therefore, (3.3) can be written as

$$E\left|1-\rho^{-1}\frac{S_n}{n}\right|, \tag{3.4}$$

where S_n is binomially distributed with parameters n and ρ. By the Central Limit Theorem, the expression in (3.4) is asymptotic to $\sqrt{2(1-\rho)/(\pi\rho n)}$ as $n\to\infty$. Using this in (3.2) leads to the first statement in the theorem, and in particular, to

$$E\big[|X|\wedge n\big]\ge C\sqrt{n} \tag{3.5}$$

for some constant $C>0$. If $E|X|^{\frac{1}{2}}<\infty$, the dominated convergence theorem applied to the sequence

$$X_n=\frac{|X|\wedge n}{\sqrt{n}}\le |X|^{\frac{1}{2}}$$

would imply

$$E\frac{|X|\wedge n}{\sqrt{n}}\to 0$$

as $n\to\infty$, which would contradict (3.5).

The case of Problem C is similar. The analogue of (3.2) is that for the distributions μ_1,μ_2 of point processes on R^1 and for $t>0$,

$$\left\|\frac{1}{t}\int_0^t (T^s\mu_1)\,ds-\frac{1}{t}\int_0^t (T^s\mu_2)\,ds\right\|\le \frac{2}{t}E\big[|X|\wedge t\big].$$

The analogue of (3.4) is

$$E\left|1-\frac{\Pi[0,t]}{t}\right|,$$

which is asymptotic to $\sqrt{2/(\pi t)}$ as $t\to\infty$. ■

Remarks. (a) My original proof of a somewhat weaker result was based on the intuition explained at the beginning of this section, and was a bit clumsy. I thank Hermann Thorisson for pointing out that the shift-coupling inequality (3.2) would probably be useful in getting negative results. He was right.

(b) One could try a similar approach in higher dimensions. The analogue of (3.5) is then

$$n^d-E\prod_{i=1}^d\big(n-|X_i|\big)^+\ge Cn^{\frac{d}{2}},$$

where $X=(X_1,\ldots,X_d)$. This does not give a contradiction for $d>1$ even if X is bounded. The intuitive argument given at the beginning of this section also relies on being in one, or possibly two, dimensions. This raises the question of finding a necessary condition for the tail behavior of X in higher dimensions. In Holroyd and Liggett (2001), we show that any solution to Problem A in two dimensions satisfies $E|X|=\infty$. We have no necessary condition for $d > 2$, other than the obvious one that $|X|$ must be at least as large as the distance to the nearest 1 in η.

(c) In general, it is not possible to achieve equality in (3.2) for all n, or even for $n = 1, 2$. For example, take $\rho = \frac{1}{2}$. Then (3.4) equals 1 for $n = 1$, and equals $\frac{1}{2}$ for $n = 2$. If equality held in (3.2) for $n = 1, 2$, it would follow that

$$E\big(|X| \wedge 1\big) = \frac{1}{2}, \quad E\big(|X| \wedge 2\big) = \frac{1}{2},$$

and hence that $P\big(|X| \geq 2\big) = 0$, which is impossible.

(d) The same proof gives a similar result if we ask only that $\{\eta(X + k), k \in Z^1\}$ be independent with

$$P\big(\eta(X + k) = 1\big) = \begin{cases} \rho & \text{if } k \neq 0, \\ \lambda & \text{if } k = 0 \end{cases}$$

for some $\lambda > \rho$.

4. Sufficient Conditions in One Dimension

We begin with a construction that works directly only if $\rho = \frac{1}{2}$. By the time Proposition 2.2 is used to extend to smaller values of ρ, however, these solutions do not scale in the way they would have to in order to give a solution to Problem C. Therefore we later give a more general construction that works directly whenever ρ is the reciprocal of an integer. Both constructions are motivated by Theorem 3.1 as follows. The best known quantity related to mean zero random walks that has tail behavior just barely satisfying the conclusion of Theorem 3.1 is the length of an excursion between visits to a given site. Therefore, a reasonable choice of X might be related to such an excursion.

The key to the first construction is a symmetry result connected with excursions. Let

$$S_k = S_k(\eta) = \sum_{i=0}^{k-1} \eta(i).$$

Then $2S_k - k$ is a mean zero integer valued random walk if η has distribution $\nu_{1/2}$.

Proposition 4.1. *Let*

$$\tau = \tau(\eta) = \min\left\{k \geq 1 : S_k = \frac{k}{2}\right\} = \min\left\{k \geq 1 : 2S_k - k = 0\right\}.$$

If η has distribution $\nu_{1/2}$, then so does ζ, where

$$\zeta(i) = \eta(\tau - i - 1).$$

Proof. Note first that τ is finite a.s., since $2S_k - k$ is recurrent, and τ is even, since $\tau = 2S_\tau$. If η has distribution $\nu_{1/2}$, then conditional on $\{\tau = 2k\}$, the distribution of η is the following:

$\{\eta(i), i \notin [0, 2k-1]\}$ are i.i.d. Bernoulli with parameter $\frac{1}{2}$, and independently of them, the vector $(\eta(0), ..., \eta(2k-1))$ is chosen uniformly from configurations that satisfy

$$S_{2k} = k, \quad \text{and} \quad S_{2i} \neq i \;\forall\; 0 < i < k. \tag{4.2}$$

Since $S_{2k} = S_{2i} + (S_{2k} - S_{2i})$, if $S_{2k} = k$, then $S_{2i} \neq i$ if and only if $S_{2k} - S_{2i} \neq k - i$. It follows that the set of configurations $(\eta(0), ..., \eta(2k-1))$ that satisfy (4.2) is invariant under the mapping

$$(\eta(0), ..., \eta(2k-1)) \quad \rightarrow \quad (\eta(2k-1), ..., \eta(0)).$$

So, the distributions of η and ζ are the same when conditioned on $\{\tau = 2k\}$, and it follows that the unconditional distributions are the same as well. ■

With τ defined as in the statement of Proposition 4.1, let

$$X = \begin{cases} 0 & \text{if } \eta(0) = 1, \\ \tau - 1 & \text{if } \eta(0) = 0. \end{cases} \tag{4.3}$$

Theorem 4.4. *If $\rho = \frac{1}{2}$, the X defined in (4.3) gives a solution to Problem A that satisfies $X \geq 0$ and*

$$\sup_{n \geq 1} \sqrt{n} P(X \geq n) < \infty.$$

Proof. Take η with distribution $\nu_{1/2}$ and define τ and ζ as in the statement of Proposition 4.1. Note that $\eta(X) = 1$. This is obvious if $\eta(0) = 1$. If $\eta(0) = 0$, then $2S_1 - 1 < 0$, and hence, since the random walk $2S_k - k$ takes only steps of size one, $2S_k - k < 0$ for all $k < \tau$. In particular, $2S_{\tau-1} < \tau - 1$, so

$$\eta(X) = \eta(\tau - 1) = S_\tau - S_{\tau-1} > \frac{\tau}{2} - \frac{\tau - 1}{2} = \frac{1}{2},$$

which implies $\eta(X) = 1$ in this case as well.

Now we need to check that $\big\{\eta(X+k), k \neq 0\big\}$ are i.i.d. Bernoulli with parameter $\frac{1}{2}$. For $0 < j_1 < \cdots < j_l$ and $0 < k_1 < \cdots < k_m$, let

$$A = A(j_1, ..., j_l; k_1, ..., k_m) = \big\{\eta(X - j_i) = 1 \text{ and } \eta(X + k_i) = 1 \;\forall i\big\}.$$

We need to show that

$$P\big(A\big) = \frac{1}{2^{l+m}}. \tag{4.5}$$

To see this, write

$$\begin{aligned} P(A) =& P(A, \eta(0) = 1) + P(A, \eta(0) = 0) \\ =& P(\eta(-j_i) = 1 \text{ and } \eta(k_i) = 1 \;\forall i, \eta(0) = 1) \\ &+ P(\eta(\tau - 1 - j_i) = 1 \text{ and } \eta(\tau - 1 + k_i) = 1 \;\forall i, \eta(\tau - 1) = 1) \\ =& \frac{1}{2^{l+m+1}} + P(\zeta(j_i) = 1 \text{ and } \zeta(-k_i) = 1 \;\forall i, \zeta(0) = 1) \\ =& \frac{1}{2^{l+m}}, \end{aligned}$$

where the final equality comes from Proposition 4.1.

Next we compute the distribution of X: $P(X=0)=P(\eta(0)=1)=\frac{1}{2}$, and

$$P(X=2k-1)=P(\tau=2k,\eta(0)=0)=\frac{1}{2}P(\tau=2k)=\frac{1}{k}\binom{2k-2}{k-1}\frac{1}{4^k},\quad k\geq 1. \tag{4.6}$$

The last equality can be found on page 180 of Fristedt and Gray (1997), for example. Since this is asymptotic to a constant multiple of $k^{-3/2}$, the result follows. ■

Applying Proposition 2.1 now gives a construction for $\rho>\frac{1}{2}$. If $\rho<\frac{1}{2}$, one can apply Proposition 2.2 and use the fact that a geometric sum of i.i.d. random variables with tails of order $n^{-1/2}$ has tails of the same order. Specifically, using the notation of Proposition 2.2 (with the ρ there being $\frac{1}{2}$ and the λ there being the current ρ),

$$\begin{aligned}P\big(X_0+\cdots+X_T\geq n\big)&=\sum_{k=0}^{\infty}P\big(X_0+\cdots+X_k\geq n\big)P\big(T=k\big)\\&\leq\sum_{k=0}^{\infty}P\big(T=k\big)(k+1)P\Big(X_0\geq\frac{n}{k+1}\Big)\\&\leq\sum_{k=0}^{\infty}P\big(T=k\big)(k+1)c\sqrt{\frac{k+1}{n}}=cn^{-\frac{1}{2}}E(T+1)^{\frac{3}{2}},\end{aligned}$$

provided that $P(X_0\geq t)\leq ct^{-\frac{1}{2}}$.

This gives a construction with the right tail behavior for each ρ. However, by (4.6), the X_0 above is in the domain of attraction of a one sided stable law of index $\frac{1}{2}$. Therefore, the solution to Problem A given by this construction is of order ρ^{-2} as $\rho\to 0$. It would need to be of order ρ^{-1} to be of use in solving Problem C.

We now turn to a construction that scales correctly as $\rho\downarrow 0$. It again provides a solution with tails that are close to optimal – compare with Theorem 3.1. Fix an integer $N\geq 2$, and let

$$I_k=\big[(k-1)N,kN\big)\cap Z^1,$$

and

$$U_k=\sum_{i\in I_k}\eta(i).$$

Put

$$\sigma=\min\{k\geq 1:U_1+\cdots+U_k\geq k\},$$

and then let $X=$ the location of the l'th nonnegative point in η if $\sigma=l$:

$$\sum_{0\leq i<X}\eta(i)=l-1\quad\text{and}\quad\sum_{0\leq i\leq X}\eta(i)=l.$$

We state and prove the following theorem and its corollary in a context that is more general than product measures or Poisson processes.

Theorem 4.7. *Suppose μ is a translation invariant and ergodic probability measure on $\{0,1\}^{Z^1}$ with density $1/N$: $\mu\{\eta : \eta(k) = 1\} = 1/N$. Let μ^* be the conditional measure $\mu^*(\cdot) = \mu(\cdot \mid \eta(0) = 1\}$. If η has distribution μ and X is defined as above, then $\eta(X + \cdot)$ has distribution μ^*.*

Proof. First we will observe that $\sigma < \infty$ a.s., and hence that X is well defined. To do so, let

$$W_k = U_1 + \cdots + U_k.$$

Then $\{W_k - k, k \geq 0\}$ is a mean zero, integer valued, random walk with stationary ergodic increments. Dekking (1982) proves that such a walk is recurrent in the sense that

$$P(W_k - k = 0 \text{ i.o.}) = 1. \tag{4.8}$$

This implies that $\sigma < \infty$ a.s. (See Theorem 3 in that paper; there are also references there to earlier versions of that result.)

Let K be a finite subset of $Z^1 \backslash \{0\}$. We need to check that

$$P\big(\eta(X + k) = 1 \ \forall\ k \in K\big) = P\big(\eta(k) = 1 \ \forall\ k \in K \mid \eta(0) = 1\big).$$

In fact, we will prove a stronger result. For any integer k let $\{k\} = k \mod N$, so that $\{k\} \in \{0, ..., N-1\}$. We will show that for each $i \in \{0, ..., N-1\}$,

$$P\big(\eta(X + k) = 1 \ \forall\ k \in K \mid \{X\} = i\big) = P\big(\eta(k) = 1 \ \forall\ k \in K \mid \eta(0) = 1\big). \tag{4.9}$$

This is immediate for $i = 0$, since, as we will now show, $\{X\} = 0$ is equivalent to $\eta(0) = 1$, and in this case, $X = 0$. To see this, note that $\eta(0) = 1$ implies $\sigma = 1$, which implies that X is the first nonnegative point in η, which is the origin. Conversely, if $\eta(0) = 0$, then there are two cases. If $\sigma = 1$, then clearly $\{X\} \geq 1$. If $\sigma = l > 1$, then $W_{l-1} < l - 1$, so the lth one in $\{0, 1, ...\}$ cannot be the first one in I_l. This completes the verification of (4.9) in case $i = 0$.

It remains to check (4.9) for other values of i. To do so, take $1 \leq i \leq N-1, l \geq 1$, $m \geq 0$, $K' = K \cup \{0\}$, and write

$$\begin{aligned}
&P\big(\eta(X + k) = 1 \ \forall\ k \in K, \sigma = l, \{X\} = i, W_{l-1} = m\big) \\
&\quad = P\big(\eta((l-1)N + i + k) = 1 \ \forall\ k \in K, \sigma = l, X = (l-1)N + i, W_{l-1} = m\big) \\
&\quad = P\big(\eta((l-1)N + i + k) = 1 \ \forall\ k \in K', W_j < j \ \forall\ 1 \leq j < l, \\
&\qquad\qquad W_{l-1} = m, \eta((l-1)N) + \cdots + \eta((l-1)N + i - 1) = l - m - 1\big).
\end{aligned} \tag{4.10}$$

Shifting the indexes by $(l-1)N + i$ and using the translation invariance of μ gives the following expression for the right side of (4.10):

$$\begin{aligned}
&P\big(\eta(k) = 1 \ \forall\ k \in K', \eta(-(l-1)N - i) + \cdots + \eta((j - l + 1)N - i - 1) < j \ \forall\ 1 \leq j < l, \\
&\qquad \eta(-(l-1)N - i) + \cdots + \eta(-i-1) = m, \eta(-i) + \cdots + \eta(-1) = l - m - 1\big) \\
&= P\big(\eta(k) = 1 \ \forall\ k \in K', \eta((j - l + 1)N - i) + \cdots + \eta(-i-1) > m - j \ \forall\ 1 \leq j < l, \\
&\qquad \eta(-(l-1)N - i) + \cdots + \eta(-i-1) = m, \eta(-i) + \cdots + \eta(-1) = l - m - 1\big) \\
&= P\big(\eta(k) = 1 \ \forall\ k \in K', V^i_j - j > -n \ \forall\ 0 \leq j < l - 1, V^i_{l-1} - (l-1) = -n, \\
&\qquad \eta(-i) + \cdots + \eta(-1) = n\big),
\end{aligned}$$

where $n = l - m - 1$ and $V_j^i = \eta(-jN - i) + \cdots + \eta(-i-1)$. If $l = 1$, then the only contributions to this sum are for $m = 0$, and hence $n = 0$. If $l \geq 2$, then contributions are possible for $0 \leq m < l - 1$, and hence for $1 \leq n \leq l - 1$. The sum on l of the above expression for fixed $n \geq 0$ is

$$P\big(\eta(k) = 1 \ \forall\ k \in K', V_j^i - j = -n \text{ for some } j \geq n, \eta(-i) + \cdots + \eta(-1) = n\big). \tag{4.11}$$

We have used here the fact that $V_j^i - j$ can only move down one step at a time, so that prior to the first hitting time of $-n$, it lies strictly above $-n$.

We will check below that

$$P\big(V_j^i - j = -n \text{ for some } j \geq n \mid \eta(0) = 1, \eta(-i) + \cdots + \eta(-1) = n\big) = 1 \tag{4.12}$$

for all values of n and i for which the conditioning event has positive probability.

Once we do this, it will follow from (4.10) and (4.11) that

$$\begin{aligned} &P\big(\eta(X + k) = 1 \ \forall\ k \in K, \{X\} = i, W_{\sigma-1} = \sigma - n - 1\big) \\ &\qquad\qquad = P\big(\eta(k) = 1 \ \forall\ k \in K', \eta(-i) + \cdots + \eta(-1) = n\big). \end{aligned} \tag{4.13}$$

Summing (4.13) on n then gives

$$P\big(\eta(X + k) = 1 \ \forall\ k \in K, \{X\} = i\big) = P\big(\eta(k) = 1 \ \forall\ k \in K'\big). \tag{4.14}$$

When $K = \emptyset$, this gives

$$P\big(\{X\} = i\big) = P\big(\eta(0) = 1\big). \tag{4.15}$$

Combining (4.14) (for general K) with (4.15) leads to (4.9) for $1 \leq i \leq N - 1$.

It remains to prove (4.12). To do so, return to the identity one gets by equating the sum on l of the left side of (4.10) and (4.11), but now with $K = \emptyset$:

$$\begin{aligned} &P\big(\{X\} = i, W_{\sigma-1} = \sigma - n - 1\big) \\ &\quad = P\big(\eta(0) = 1, V_j^i - j = -n \text{ for some } j \geq n, \eta(-i) + \cdots + \eta(-1) = n\big) \\ &\quad \leq P\big(\eta(0) = 1, \eta(-i) + \cdots + \eta(-1) = n\big). \end{aligned} \tag{4.16}$$

Note that the right and left sides of (4.16) agree for $i = 0$ as well. Now sum (4.16) for $0 \leq i \leq N - 1$ and $n \geq 0$. The result is

$$1 \leq NP\big(\eta(0) = 1\big). \tag{4.17}$$

Since this is an equality by assumption, it must be the case that equality holds throughout (4.16) for all choices of n and i. But this is just (4.12). ■

Remark. Note that in the proof above, we used $P\big(\eta(k) = 1\big) \geq \frac{1}{N}$ to show that $\sigma < \infty$ a.s. at the beginning, and used $P\big(\eta(k) = 1\big) \leq \frac{1}{N}$ in the final step of the argument.

Next we prove the analogous result for the continuum problem.

Corollary 4.18. *Suppose* Π *is a stationary ergodic simple point process on* R^1 *whose intensity measure is Lebesgue measure. Let* $\sigma = \min\{k \geq 1, k \in Z^1 : \Pi([0,k) \geq k\}$. *If* $\sigma = l$, *let* X *be the* l*th positive point in* Π, *i.e., the one that satisfies* $\Pi([0,X) = l-1$ *and* $\Pi([0,X] = l$. *Then* $\Pi(X+\cdot)$ *has the corresponding Palm distribution.*

Proof. For an integer $N \geq 2$, define $\eta_N \in \{0,1\}^{Z^1}$ by

$$\eta_N(k) = \Pi\Big(\Big(\frac{k}{N}, \frac{k+1}{N}\Big]\Big) \wedge 1.$$

Then the distribution of η_N satisfies the assumptions of Theorem 4.7. Let X_N be the solution to Problem A for η_N constructed prior to the statement of that theorem, and let σ_N the corresponding stopping time. Since Π is simple, the probability that Π has two points a distance $\leq \frac{1}{N}$ from each other within a fixed compact set tends to zero as $N \to \infty$, so

$$\sigma_N \to \sigma \quad \text{and} \quad \frac{X_N}{N} \to X$$

a.s. Furthermore, for any interval I,

$$\Pi(I) = \lim_{N\to\infty} \sum_{k \in NI} \eta_N(k).$$

Therefore, if I is an interval such that $\Pi(X + \partial I) = 0$ a.s. (this rules out at most a deterministic countable set of endpoints of these intervals – those that are in the group generated by 0 and the possible values of spacings of Π),

$$\begin{aligned}\Pi\big(X+I\big) &= \lim_{N\to\infty} \sum_{k\in NX+NI} \eta_N(k)\\ &= \lim_{N\to\infty} \sum_{k\in X_N+NI} \eta_N(k)\\ &= \lim_{N\to\infty} \sum_{k\in NI} \eta_N(X_N+k).\end{aligned}$$

The last sum above has the distribution of

$$\Big(\sum_{k\in NI} \eta_N(k) \,\Big|\, \eta_N(0) = 1\Big)$$

by Theorem 4.7. One property of the Palm distribution is that it is the limiting distribution as $\epsilon \to 0$ of Π conditioned on $\Pi[0,\epsilon) \geq 1$. (See Theorem 5.4 of Chapter 8 of Thorisson (2000), for example.) Since $\Pi[0,\frac{1}{N}) \geq 1$ is equivalent to $\eta_N(0) = 1$, and the Palm process is also simple, the result follows. ■

Now we specialize to the case of Problems A and C to determine the tail behavior of X. For Problem A with $\rho = \frac{1}{N}$ for some integer $N > 1$, we use the construction described prior to the statement of Theorem 4.7. For ρ falling between two such reciprocals, we apply that construction to the product measure with density equal to the reciprocal of an integer just below it, and then use Proposition 2.1. For Problem C, we use the construction described in the statement of Corollary 4.18.

Theorem 4.19. *The construction described above has the following property: In the case of Problem A,*

$$\lim_{t\to\infty}\frac{E\big(|N^{-1}X|\wedge t\big)}{\sqrt{t}}\le 8\sqrt{\frac{1-N^{-1}}{2\pi}},$$

where the integer N is determined by $\frac{1}{N}\le\rho<\frac{1}{N-1}$. In the case of Problem C,

$$\lim_{t\to\infty}\frac{E\big(|X|\wedge t\big)}{\sqrt{t}}\le 8\sqrt{\frac{1}{2\pi}}.$$

Furthermore, letting X_ρ be the solution for ν_ρ, we have the following uniform bound on the tails:

$$\sup_{t>0,0<\rho<1}\sqrt{t}P\big(\rho X_\rho\ge t\big)<\infty.$$

Proof. By Proposition 2.1, we may assume that we are considering either Problem A with $\rho=1/N$ for some positive integer N, or Problem C. It is enough to carry out the computations in the former case, since the latter is similar.

Note first that $X\le N\sigma$. Since the random walk $U_1+\cdots+U_n-n$ is left continuous many hitting distributions can be calculated explicitly. For example, let

$$\sigma^*=\min\{k\ge 1: U_1+\cdots+U_k-k>0\}\ge\sigma.$$

According to Section 17 of Spitzer (1976), the generating function of σ^* is given by

$$1-Eu^{\sigma^*}=\left(1-\frac{1}{N}\right)^{-N}\frac{r(u)}{1-r(u)}\frac{1-u}{u},\tag{4.20}$$

where $0<r(u)<1$ satisfies

$$\frac{r(u)}{u}=\left[1-\frac{1-r(u)}{N}\right]^N,\quad 0<u<1.\tag{4.21}$$

Take logarithms in (4.21), and use the Taylor expansion of the logarithm about 1 to check that

$$\lim_{u\uparrow 1}\frac{1-r(u)}{\sqrt{1-u}}=\sqrt{2\left(1-\frac{1}{N}\right)^{-1}}.$$

Using this in (4.20) gives

$$\lim_{u\uparrow 1}\frac{1-Eu^{\sigma^*}}{\sqrt{1-u}}=\frac{1}{\sqrt{2}}\left(1-\frac{1}{N}\right)^{\frac{1}{2}-N}.\tag{4.22}$$

Now write the numerator of the left side of (4.22) as

$$1-Eu^{\sigma^*}=\frac{1-u}{u}\sum_{k=1}^\infty P\big(\sigma^*\ge k\big)u^k.\tag{4.23}$$

Applying a Tauberian Theorem (see, for example, Theorem 108 on page 166 of Hardy (1956)) and (4.22) gives

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} P\big(\sigma^* \geq k\big) = \sqrt{\frac{2}{\pi}} \left(1 - \frac{1}{N}\right)^{\frac{1}{2}-N}.$$

It remains to write

$$E\big(|N^{-1}X| \wedge n\big) \leq E(\sigma \wedge n) \leq E(\sigma^* \wedge n) = \sum_{k=1}^{n} P\big(\sigma^* \geq k\big)$$

and note that

$$\left(1 - \frac{1}{N}\right)^{-N} \leq 4$$

for $N \geq 2$ to complete the proof of the first statement of the theorem.

To get the final uniformity statement, we proceed as follows. By Proposition 2.1, we may take $\rho = 1/N$, and ask for uniformity in N. Let $r_N(u)$ be the solution of (4.30), and put

$$r(u) = \lim_{N\to\infty} r_N(u).$$

Then $r(u)$ satisfies

$$\frac{r(u)}{u} = e^{r(u)-1} \tag{4.24}$$

by (4.21). Since $(1 - \frac{x}{N})^N$ is increasing in N for $0 \leq x \leq 1$, it follows that

$$r_N(u) \leq r(u), \quad N \geq 1, 0 < u < 1. \tag{4.25}$$

By (4.24),

$$r(u) \sim e^{-1}u \text{ as } u \downarrow 0 \quad \text{and} \quad 1 - r(u) \sim \sqrt{2(1-u)} \text{ as } u \uparrow 1. \tag{4.26}$$

Combining (4.25) and (4.26), it follows that there is a constant C so that

$$\frac{r_N(u)}{1 - r_N(u)} \leq C \frac{u}{\sqrt{1-u}}$$

for all $N \geq 1$ and $0 < u < 1$. Using this is (4.20) we see that

$$1 - Eu^{\sigma^*} \leq C\sqrt{1-u}.$$

Using this and (4.23),

$$P\big(\sigma^* \geq n\big) \sum_{k=1}^{n} u^k \leq \sum_{k=1}^{\infty} P\big(\sigma^* \geq k\big) u^k \leq \frac{C}{\sqrt{1-u}},$$

and then by summing the series on the left,

$$P\big(\sigma^* \geq n\big)u(1-u^n) \leq C\sqrt{1-u}.$$

Taking $u = 1 - \frac{1}{n}$ gives

$$P\big(\sigma^* \geq n\big) \leq \frac{C}{\sqrt{n}}$$

for a new constant C. Since

$$\rho X_\rho \leq \sigma \leq \sigma^*,$$

the result follows. ■

Remarks. (a) When $\mu = \nu_{1/2}$, the constructions in Theorems 4.4 and 4.7 are identical.

(b) In the case of Problem A with $\rho =$ the reciprocal of an integer, and in the case of Problem C, the construction in Theorem 4.7 does not require additional randomization. For other values of ρ it does, through the use of Proposition 2.1. One might ask whether there is a construction for arbitrary ρ with good tails that does not use additional randomization.

5. Extensions

In this section, we summarize what can be said about the analogues of Problems A and B for more general distributions, and then apply this to stationary renewal measures. Here μ will be a translation invariant ergodic probability measure on $\{0,1\}^{Z^1}$, and μ^* will be the corresponding conditional measure:

$$\mu^*(\cdot) = \mu(\cdot \mid \eta(0) = 1).$$

We consider random (η, X) so that η has distribution μ and $T^X\eta$ has distribution μ^*. Using the proof of Theorem 3.1, we obtain the following:

Theorem 5.1. *If η has distribution μ and $T^X\eta$ has distribution μ^*, then*

$$2E\big(|X| \wedge n\big) \geq E\left|n - \rho^{-1}\sum_{k=1}^{n}\eta(k)\right|, \tag{5.2}$$

where $\rho = P\big(\eta(0) = 1\big)$.

When $P\big(\eta(0) = 1\big)$ is the reciprocal of an integer, Theorem 4.7 provides a construction of X with the desired property. To deal with other densities, one might guess is that one could use the approach in Propostion 2.1: try X corresponding to ζ, where ζ is obtained from η by independent thinning. It is not clear that this provides a solution, though. Here is another approach that does. Let η have distribution μ and U be a random variable independent of η that is uniform on $[0,1]$. Define a point process on the line by

$$\Pi(A) = \sum_{k \in A+U} \eta(k).$$

Now use Corollary 4.18 (modified to account for the fact that the intensity measure of Π is a multiple of Lebesgue measure). For simplicity, in the next statement, we consider only measures wth density $\rho = \frac{1}{N}$ for some integer $N \geq 2$. The following restates Theorem 4.7.

Theorem 5.3. *Suppose η has distribution μ with density $\frac{1}{N}$, where $N \geq 2$. Then there exists a random variable $X \geq 0$ so that $T^X\eta$ has distribution μ^* and*

$$X \leq N \min\left\{k \geq 1 : \sum_{i=0}^{Nk-1} \eta(i) \geq k\right\}.$$

As an example of the application of these results, we consider stationary renewal measures μ with density $\rho = \frac{1}{2}$. Let $f(n), n \geq 1$, be a probability density of mean 2, and let

$$F(n) = \sum_{k=n}^{\infty} f(k)$$

be the corresponding tail probabilities. Let $\{Y_k, k \geq 0\}$ be a sequence of independent random variables with densities

$$P\big(Y_0 = n\big) = \frac{1}{2}F(n) \quad \text{and} \quad P\big(Y_k = n\big) = f(n)$$

for $k, n \geq 1$. By the renewal property, if η has distribution μ, then

$$\{k \geq 1 : \eta(k) = 1\} =_d \{Y_0, Y_0 + Y_1, Y_0 + Y_1 + Y_2, ...\}. \tag{5.4}$$

Here $=_d$ means equality in distribution.

Now we will compute the right side of (5.2) for even n. Since $n - 2\sum_{1\leq k\leq n} \eta(k)$ has mean zero,

$$\begin{aligned} E\left|n - 2\sum_{k=1}^{n} \eta(k)\right| &= 2E\left[2\sum_{k=1}^{n} \eta(k) - n, 2\sum_{k=1}^{n} \eta(k) > n\right] \\ &= 2\sum_{m=1}^{\infty} P\left(2\sum_{k=1}^{n} \eta(k) - n \geq m\right) \\ &= 4\sum_{m=\frac{n}{2}}^{\infty} P\left(\sum_{k=0}^{m} Y_k \leq n\right) \\ &= 4ER\left(n - \sum_{k=0}^{\frac{n}{2}} Y_k\right), \end{aligned} \tag{5.5}$$

where

$$R(x) = \sum_{m=0}^{\infty} P\left(\sum_{k=1}^{m} Y_k \leq x\right)$$

is the renewal function. In the third equality of (5.5) we have used (5.4). The extra factor of 2 comes from the fact that the terms on the left corresponding to even m agree with the terms corresponding to odd m. By the renewal theorem,

$$\lim_{x\to\infty} \frac{R(x)}{x} = \frac{1}{2}.$$

It follows from this, (5.5) and the fact that $n-\sum_{1\le k\le n/2} Y_k$ has mean zero that

$$\lim_{n\to\infty}\frac{E\Big|n-2\sum_{1\le k\le n}\eta(k)\Big|}{E\Big|n-\sum_{1\le k\le n/2}Y_k\Big|}=1. \tag{5.6}$$

We turn now to the tails of the solution X provided by Theorem 5.3. Using (5.4) again, write

$$\begin{aligned} P\big(X\ge 2n\big) &\le P\Big(\sum_{i=1}^{2k}\eta(i)<k\ \forall\ 1\le k<n\Big)\\ &= P\Big(\sum_{i=0}^{k-1}Y_i>2k\ \forall\ 1\le k<n\Big)\\ &= P\Big(\sigma(Y_0-2)\ge n-1, Y_0\ge 3\Big), \end{aligned} \tag{5.7}$$

where

$$\sigma(m)=\min\Big\{k\ge 1:\sum_{i=1}^{k}Y_i-2k=-m\Big\}$$

is the hittiing time of $-m$ for the recurrent random walk $\sum_{1\le i\le k}(Y_i-2)$. Since this random walk is left continuous, the distribution of $\sigma(m)$ can be computed explicitly for $m\ge 1$. To do so, let

$$\phi(u)=Eu^{\sigma(1)}$$

be the generating function for $\sigma(1)$. Then the generating function of $\sigma(m)$ is given by

$$Eu^{\sigma(m)}=\big[\phi(u)\big]^m,\quad m\ge 1.$$

Running the random walk for one step leads to the equation

$$u\psi\big[\phi(u)\big]=1, \tag{5.8}$$

where

$$\psi(u)=Eu^{Y_1-2}=\sum_{j=1}^{\infty}f(j)u^{j-2}.$$

Multiplying (5.7) by u^n and summing leads to

$$\sum_{n=1}^{\infty}P\big(X\ge 2n\big)u^n\le\frac{u}{2}\Big[f(1)+\frac{\phi(u)}{1-\phi(u)}\Big].$$

Letting $u = 1 - \frac{1}{n}$ in this inequality gives

$$P\big(X \geq 2n\big) \leq \frac{C}{n\left[1 - \phi\left(1 - \frac{1}{n}\right)\right]} \tag{5.9}$$

for some constant C – see the argument at the end of the proof of Theorem 4.19.

We now specialize to the case in which Y_1 is in the domain of normal attraction of a stable law of index $\alpha \in (1, 2]$. In this case, combining (5.2) and (5.6), we see that any X for which $T^X\eta$ has distribution μ^* satisfies

$$\inf_{n\geq 1} \frac{E\big(|X| \wedge n\big)}{n^{1/\alpha}} > 0. \tag{5.10}$$

On the other hand, since $\psi(u) - 1 \sim c(1-u)^\alpha$ as $u \uparrow 1$, (5.8) implies that $1 - \phi(u) \sim c'(1-u)^{1/\alpha}$ as $u \uparrow 1$. Here c and c' are positive constants. Therefore, the solution X provided by Theorem 5.3 satisfies

$$\sup_{n\geq 1} n^{1-(1/\alpha)} P\big(X \geq n\big) < \infty \tag{5.11}$$

by (5.9). Since

$$E\big(|X| \wedge n\big) = \sum_{k=1}^{n} P\big(|X| \geq k\big),$$

(5.10) and (5.11) are essentially sharp.

Appendix

The solutions to Problems A and C obtained by Thorisson and others are quite different from those given in Section 4. They are based on the method of "common components". Here is the basic idea. We want to find a shift coupling of two measures μ_1 and μ_2, i.e., a construction (η, X) so that η has distribution μ_1 and T^X has distribution μ_2. Start by considering the common component $\mu_1 \wedge \mu_2$, i.e., the measure whose Radon-Nikodym derivative with respect to $\mu_1 + \mu_2$ is the pointwise minimum of the Radon-Nikodym derivatives of μ_1 and μ_2 with respect to $\mu_1 + \mu_2$. On this common part of the space, take $X = 0$, and consider the remaining parts of the measures: $\mu_1 - \mu_1 \wedge \mu_2$ and $\mu_2 - \mu_1 \wedge \mu_2$. These are mutually singular, but if one of them is shifted, the resulting measures may again have a common component. This procedure can be repeated indefinitely. In this section, we determine what the tails of the resulting X are in the case of Problem A. We do not consider general shifts, which would lead to very difficult computations, but rather shift by successive powers of 2.

Theorem A.1. *The construction via common components with shifts that are successive powers of 2 gives a solution X to Problem A that satisfies $X \geq 0$ and*

$$0 < \inf_{n\geq 1} \big[\log n P(X \geq n)\big] \leq \sup_{n\geq 1} \big[\log n P(X \geq n)\big] < \infty.$$

Proof. Let ν_ρ^* be the conditional measure $\nu_\rho\big(\cdot\,|\,\eta(0)=1\big)$. We need to find a joint distribution of (η, X) so that η has distribution ν_ρ and $T^X\eta$ has distribution ν_ρ^*. We will take X to have only values of the form 2^n-1 for $n\geq 0$, and will find a sequence of measures $\{\sigma_n, n\geq 0\}$ so that

$$\nu_\rho=\sum_{n=0}^\infty \sigma_n \quad \text{and} \quad \nu_\rho^*=\sum_{n=0}^\infty T^{2^n-1}\sigma_n. \tag{A.2}$$

The desired joint distribution is given by

$$P(\eta\in A, X=2^n-1)=\sigma_n(A), \quad n\geq 0, \tag{A.3}$$

since then

$$\nu_\rho(A)=\sum_{n=0}^\infty \sigma_n(A)=P(\eta\in A)$$

and

$$\nu_\rho^*(A)=\sum_{n=0}^\infty \sigma_n(T^{-2^n+1}A)=\sum_{n=0}^\infty P(T^{2^n-1}\eta\in A, X=2^n-1)=P(T^X\eta\in A).$$

The σ_n's will be defined recursively. If we want the $n=0$ terms in (A.2) to contribute as much as possible to the sum, we should choose $\sigma_0=\nu_\rho\wedge\nu_\rho^*$. Suppose now that we have chosen $\sigma_0, ..., \sigma_{m-1}$, and want to choose σ_m. Rewrite (A.2) as

$$\nu_\rho-\sum_{n=0}^{m-1}\sigma_n=\sum_{n=m}^\infty \sigma_n \quad \text{and} \quad T^{-2^m+1}\nu_\rho^*-\sum_{n=0}^{m-1}T^{2^n-2^m}\sigma_n=\sum_{n=m}^\infty T^{2^n-2^m}\sigma_n.$$

The point of writing this in this way is that the summands corresponding to $m=n$ on the right of the two expressions above are the same, so it makes sense to choose

$$\sigma_m=\alpha_m\wedge\beta_m,$$

where

$$\alpha_m=\left[\nu_\rho-\sum_{n=0}^{m-1}\sigma_n\right] \quad \text{and} \quad \beta_m=T^{-2^m+1}\nu_\rho^*-\sum_{n=0}^{m-1}T^{2^n-2^m}\sigma_n.$$

In order for this construction to work, it is enough that

$$\sum_{n=0}^\infty \sigma_n(\{0,1\}^{Z^1})=1, \tag{A.4}$$

since clearly

$$\nu_\rho\geq\sum_{n=0}^\infty \sigma_n \quad \text{and} \quad \nu_\rho^*\geq\sum_{n=0}^\infty T^{2^n-1}\sigma_n.$$

Note that by (A.3), (A.4) (once we have shown it holds) and the definition of α_n,

$$P\big(X = 2^n - 1\big) = \sigma_n(\{0,1\}^{Z^1}) \quad \text{and} \quad P(X \geq 2^n - 1) = \alpha_n(\{0,1\}^{Z^1}), \tag{A.5}$$

so that the tail behavior of X is determined by the total masses of the α_n's.

In principle, we just need to compute these measures. Each α_n and β_n is given by a particular measure on 2^{n-1} coordinates: $\{0, ..., 2^{n-1} - 1\}$ for α_n and $\{2^{n-1}, ..., 2^n - 1\}$ for β_n. The corresponding σ_n depends on the 2^n coordinates $\{-2^n + 1, ..., 0\}$. The remaining coordinates are distributed according to ν_ρ in each case. We write down a few values, giving only the values that the measures assign to the cylinder events that have positive weight:

$$\alpha_1(0) = 1 - \rho, \quad \beta_1(1) = 1 - \rho, \quad \sigma_1(01) = \begin{cases} \rho(1-\rho) & \text{if } \rho \leq \frac{1}{2}, \\ (1-\rho)^2 & \text{if } \rho \geq \frac{1}{2}, \end{cases}$$

$$\alpha_2(00) = (1-\rho)^2, \quad \alpha_2(01) = \begin{cases} 0 & \text{if } \rho \leq \frac{1}{2}, \\ (1-\rho)(2\rho - 1) & \text{if } \rho \geq \frac{1}{2}, \end{cases}$$

$$\beta_2(11) = \rho(1-\rho), \quad \beta_2(01) = \begin{cases} (1-\rho)(1-2\rho) & \text{if } \rho \leq \frac{1}{2}, \\ 0 & \text{if } \rho \geq \frac{1}{2}, \end{cases}$$

$$\sigma_2(0111) = \begin{cases} 0 & \text{if } \rho \leq \frac{1}{2}, \\ \rho^2(1-\rho)(2\rho-1) & \text{if } \frac{1}{2} \leq \rho \leq \frac{2}{3}, \\ \rho^2(1-\rho)^2 & \text{if } \rho \geq \frac{2}{3}, \end{cases}$$

$$\sigma_2(0011) = \begin{cases} \rho^2(1-\rho)^2 & \text{if } \rho \leq \frac{1}{2}, \\ \rho(1-\rho)^3 & \text{if } \rho \geq \frac{1}{2}, \end{cases}$$

$$\sigma_2(0001) = \begin{cases} \rho(1-\rho)^3 & \text{if } \rho \leq \frac{1}{3}, \\ (1-\rho)^3(1-2\rho) & \text{if } \frac{1}{3} \leq \rho \leq \frac{1}{2}, \\ 0 & \text{if } \rho \geq \frac{1}{2}. \end{cases}$$

From these examples, it is clear that computation of these measures for general ρ is impractical. However, the computations simplify considerably if we take $\rho = \frac{1}{N}$ for some positive integer N. By Proposition 2.1, it is enough to carry out the remainder of the proof in this case.

So, from now on, we take $\rho = \frac{1}{N}$. We will see inductively on n that the weights that α_n and β_n put on configurations on the sites $\{0, ..., 2^{n-1} - 1\}$ and $\{2^{n-1}, ..., 2^n - 1\}$ respectively are always of the form $j\rho^k(1-\rho)^{2^{n-1}-k}$, where k is the number of ones in the configuration (and therefore $2^{n-1} - k$ is the number of zeros in the configuration), and j is an integer in the range $0 \leq j \leq 1$ in the case of α_n and $0 \leq j \leq N - 1$ in the case of β_n. (Note that for this to be true for $n = 1$ requires that $(1-\rho)/\rho$ be an integer, and it is for this reason that we take $\rho = 1/N$.) Let $f_n(j, k)$ and $g_n(j, k)$ be the number of configurations with this weight for α_n and β_n respectively. Then for $0 \leq i \leq N - 1$ and $0 \leq j \leq 1$, there are $f_n(j, l)g_n(i, k)$ configurations on $\{0, ..., 2^n - 1\}$ with l ones on the first half of this interval and k ones on the second half that have weight $j\rho^{k+l}(1-\rho)^{2^n-k-l}$ under α_n and weight $i\rho^{k+l}(1-\rho)^{2^n-k-l}$

under β_n. So, σ_n assigns weight $(i \wedge j)\rho^{k+l}(1-\rho)^{2^n-k-l}$ to such a configuration, and hence α_{n+1} and β_{n+1} assign weights

$$(j-i)^+\rho^{k+l}(1-\rho)^{2^n-k-l} \quad \text{and} \quad (i-j)^+\rho^{k+l}(1-\rho)^{2^n-k-l}$$

to it respectively. This completes the inductive argument announced at the beginning of this paragraph, and leads to the recursions

$$\begin{aligned} f_{n+1}(u,v) &= \sum_{k+l=v,(j-i)^+=u} f_n(j,l)g_n(i,k), \\ g_{n+1}(u,v) &= \sum_{k+l=v,(i-j)^+=u} f_n(j,l)g_n(i,k). \end{aligned} \tag{A.6}$$

Now let

$$\phi_n(j) = \sum_{k=0}^{2^{n-1}} f_n(j,k)\rho^k(1-\rho)^{2^{n-1}-k}, \quad 0 \le j \le 1$$

and

$$\gamma_n(i) = \sum_{k=0}^{2^{n-1}} g_n(i,k)\rho^k(1-\rho)^{2^{n-1}-k}, \quad 0 \le i \le N-1.$$

The total masses of α_n and β_n (which are equal) can then be expressed in terms of these quantities:

$$\phi_n(1) = \alpha_n\big(\{0,1\}^{Z^1}\big) = \beta_n\big(\{0,1\}^{Z^1}\big) = \sum_{i=0}^{N-1} i\gamma_n(i). \tag{A.7}$$

The recursions in (A.6) lead to the following recursions for the ϕ's and γ's:

$$\phi_{n+1}(u) = \sum_{(j-i)^+=u} \phi_n(j)\gamma_n(i), \quad \gamma_{n+1}(u) = \sum_{(i-j)^+=u} \phi_n(j)\gamma_n(i). \tag{A.8}$$

Inductively, it follows from these that

$$\sum_{u=0}^{1} \phi_n(u) = \sum_{u=0}^{N-1} \gamma_n(u) = 1, \quad n \ge 1. \tag{A.9}$$

Identities (A.7) and (A.9) allow us to write a single recursion for the γ's:

$$\gamma_{n+1}(u) = \gamma_n(u) + \big[\gamma_n(u+1) - \gamma_n(u)\big]\tau_n, \quad 1 \le u \le N-1, \tag{A.10}$$

where we have set $\gamma_n(N) = 0$ and $\tau_n = \sum_{i=0}^{N-1} i\gamma_n(i)$.

Note that $\gamma_1(N-1) = 1/N > 0$ and $\gamma_1(0) = (N-1)/N > 0$, and (A.10) implies then that $\gamma_n(u) > 0$ for $N-n \le u \le N-1$, and $\gamma_n(u) = 0$ for $1 \le u < N-n$. Multiply (A.10) by u and sum for $0 \le u \le N-1$ to get

$$\begin{aligned}\tau_{n+1} = \sum_{u=0}^{N-1} u\gamma_{n+1}(u) =& \tau_n\Big[1 + \sum_{u=0}^{N-1} u\big[\gamma_n(u+1) - \gamma_n(u)\big]\Big] \\ =& \tau_n\Big[1 - \sum_{u=1}^{N-1} \gamma_n(u)\Big].\end{aligned} \tag{A.11}$$

It follows from (A.11) that

$$\lim_{n\to\infty} \gamma_n(u) = 0, \quad 1 \le u \le N-1. \tag{A.12}$$

To see this, note that by (A.11), $\tau_n = \sum_{u=0}^{N-1} u\gamma_n(u)$ is decreasing in n. If its limit is zero, then (A.12) is immediate, while if it is strictly positive, it would follow from (A.11) that $\sum_{u=1}^{N-1} \gamma_n(u)$ tends to zero, and this again would imply (A.12). Statement (A.12) already implies (A.4).

We next need to determine the rate of convergence in (A.12), so that we can use (A.5) to determine the size of the tails of X. To do so, sum (A.10) for $1 \le u \le N-1$, obtaining

$$\sum_{u=1}^{N-1} \gamma_{n+1}(u) = \sum_{u=1}^{N-1} \gamma_n(u) - \gamma_n(1)\tau_n. \tag{A.13}$$

Since $\gamma_n(1) = 0$ for $n \le N-2$, (A.11) and (A.13) can be used inductively to check that

$$\sum_{u=1}^{N-1} \gamma_n(u) = \frac{1}{N}, \quad \tau_n = \left[\frac{N-1}{N}\right]^n \tag{A.14}$$

for $n < N$.

Next, rewrite (A.10) in a more convenient way. Begin with

$$\frac{1}{\gamma_{k+1}(u)} - \frac{1}{\gamma_k(u)} = \frac{\gamma_k(u) - \gamma_{k+1}(u)}{\gamma_k(u)\gamma_{k+1}(u)} = \frac{\gamma_k(u) - \gamma_k(u+1)}{\gamma_k(u)\gamma_{k+1}(u)}\tau_k.$$

Summing this for $N-u \le k \le n-1$ leads to

$$\frac{1}{\gamma_n(u)} - \frac{1}{\gamma_{N-u}(u)} = \sum_{k=N-u}^{n-1} \frac{\gamma_k(u) - \gamma_k(u+1)}{\gamma_k(u)\gamma_{k+1}(u)}\tau_k. \tag{A.15}$$

Take $1 \le k \le N-2$, and use (A.10) to write

$$\frac{\gamma_{n+1}(k+1)}{\gamma_{n+1}(k)} = \frac{\gamma_n(k+1)[1-\tau_n] + \gamma_n(k+2)\tau_n}{\gamma_n(k)[1-\tau_n] + \gamma_n(k+1)\tau_n}, \tag{A.16}$$

which is an average of

$$\frac{\gamma_n(k+1)}{\gamma_n(k)} \quad \text{and} \quad \frac{\gamma_n(k+2)}{\gamma_n(k+1)},$$

and hence lies between them. When $k = N-2$, we see from this that

$$\frac{\gamma_n(k+1)}{\gamma_n(k)} \downarrow \quad \text{in } n,$$

since $\gamma_n(N) = 0$. Inductively, it follows from this and the previous observation that

$$\epsilon_k = \lim_{n\to\infty} \frac{\gamma_n(k+1)}{\gamma_n(k)} \geq 0 \tag{A.17}$$

exists and is finite.

Dividing (A.10) by $\gamma_n(u)$ and passing to the limit, it follows from (A.12) and (A.17) that

$$\lim_{n\to\infty} \frac{\gamma_{n+1}(u)}{\gamma_n(u)} = 1, \quad 1 \leq u \leq N-1. \tag{A.18}$$

Dividing (A.15) by n and passing to the limit using (A.17), (A.18), and $\tau_k \geq u\gamma_k(u)$ (which comes from keeping only one term in the definition of τ_k), we see that

$$\liminf_{n\to\infty} \frac{1}{n\gamma_n(u)} \geq u(1-\epsilon_u). \tag{A.19}$$

We will now show by induction on u that

$$\liminf_{n\to\infty} \frac{1}{n\gamma_n(u)} > 0, \quad 1 \leq u \leq N-1. \tag{A.20}$$

Since $\gamma_n(N) = 0$, $\epsilon_{N-1} = 0$. Therefore, (A.19) implies (A.20) for $u = N-1$ — this is the basis step. In fact, this argument shows that (A.20) holds for every value of u for which $\epsilon_u = 0$. If $\epsilon_u > 0$ for some u, then (A.20) for $u+1$ and (A.17) for $k = u$ imply (A.20) for u. This completes the induction argument.

Taking $u = 1$ in (A.15) and using (A.17) and (A.18) as before, we see that

$$\limsup_{n\to\infty} \frac{1}{n\gamma_n(1)} \leq (1-\epsilon_1) \sum_{i=1}^{N-1} i \prod_{j=1}^{i-1} \epsilon_j < \infty. \tag{A.21}$$

Returning to the definition of τ_n, it follows from (A.20) and (A.21) that

$$0 < \inf_n n\tau_n \leq \sup_n n\tau_n < \infty. \tag{A.22}$$

By (A.5), (A.7) and the definition of τ_n,

$$P\big(X \geq 2^n - 1\big) = \tau_n, \tag{A.22}$$

so the statement of the theorem follows from (A.21). ■

Remarks. (a) This construction does not have good scaling properties, and therefore does not say anything about Problem C. To see this, note that if $\rho = 1/N$ and X_N is the corresponding solution, the expression for τ_n in (A.14) together with (A.22) give

$$\lim_{N\to\infty} P\big(X_N \geq 2^{N-1}\big) = e^{-1}.$$

In order for a version of this construction to apply to Problem C, we would need X_N to be of order N as $N \to \infty$.

(b) The above proof simplifies considerably if $N = 2$. In this case, the total mass of α_n is $\gamma_n(1)$, which by (A.10) satisfies the recursion

$$\gamma_{n+1}(1) = \gamma_n(1)\big[1 - \gamma_n(1)\big].$$

Since $\gamma_1(1) = \frac{1}{2}$, it is then easy to check by induction that $\frac{1}{2n} \leq \gamma_n(1) \leq \frac{1}{n}$ for $n \geq 1$.

Acknowledgement. I am grateful to Ander Holroyd for many useful conversations on the topic of this paper.

References

D. J. Aldous and H. Thorisson, *Shift-coupling*, Stoch. Proc. Appl. **44** (1993), 1–14.

Dekking, *On transience and recurrence of generalized random walks*, Z. Wahrsch. Verw. Gebiete **61** (1982), 459–465.

B. Derrida, S. Goldstein, J. L. Lebowitz and E. R. Speer, *Shift equivalence of measures and the intrinsic structure of shocks in the asymmetric simple exclusion process*, J. Statist. Phys. **93** (1998), 547–571.

B. Fristedt and L. Gray, *A Modern Approach to Probability Theory*, Birkhäuser, 1997.

H. O. Georgii, *Orbit coupling*, Ann. Inst. Henri Poincaré **33** (1997), 253–268.

G. H. Hardy, *Divergent Series*, second edition, Oxford, 1956.

A. Holroyd and T. M. Liggett, *How to find an extra head: optimal random shifts of Bernoulli and Poisson random fields.*

C. Kipnis, *Central limit theorems for infinite series of queues and applications to simple exclusion*, Ann. Probab. **14** (1986), 397–408.

T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, Springer, 1999.

F. Spitzer, *Principles of Random Walk*, second edition, Springer, 1976.

H. Thorisson, *On time- and cycle stationarity*, Stoch. Proc. Appl. **55** (1995), 183–209.

H. Thorisson, *Transforming random elements and shifting random fields*, Ann. Probab. **24** (1996), 2057–2064.

H. Thorisson, *Coupling, Stationarity, and Regeneration*, Springer, 2000.

H. Thorisson, *Point-stationarity in d dimensions and Palm theory*, Bernoulli **5** (1999), 797–831.

Department of Mathematics
University of California, Los Angeles
Los Angeles CA 90095
tml@math.ucla.edu