Survival of the Contact+Voter Process on the Integers

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Abstract

We establish an upper bound on the critical value of the process obtained by superimposing the voter model on the contact process in one dimension.

1 Introduction

In [1], we used a rather special technique to prove that the basic contact process in one dimension survives if $\lambda = 2$. Durrett asked whether one could apply this technique to prove survival of the process obtained by adding to it a voter model interaction. This process has the following transition rates at site $x$:

$$
0 \to 1 \text{ at rate } (\lambda + \theta) \#(y : \eta(y) = 1, |y - x| = 1),
$$

$$
1 \to 0 \text{ at rate } 1 + \theta \#(y : \eta(y) = 0, |y - x| = 1).
$$

In this note, we work out the details of the argument in this case. Here is the result:

**Theorem 1.** The process survives if

$$
\theta = \frac{4\sqrt{2\lambda^{2} - 6\lambda^{2} - 2\sqrt{2}\lambda^{2}}}{8\lambda - 1} = 2\lambda^{2} \frac{\sqrt{\lambda} - \sqrt{2}}{2\sqrt{2\lambda} - 1}.
$$

Note that when $\theta = 0, \lambda = 2$. The proof follows pages 168–174 of [3], using the notation there.

It has three parts. They are the analogues of Proposition 4.57, Exercise 4.59 and Proposition 4.58 there. The first provides a useful expression for the time derivatives $Q(A)$ of the evolution at $t = 0$. The second gives the log-convexity of the tail probabilities of the distribution determining the initial renewal measure for the evolution, while the third uses this to prove the monotonicity that is needed to show $Q(A) \leq 0$ for all $A$.

Once these analogues are established, the proof of the theorem proceeds as in the case of the contact process. There are two ways of viewing the argument. The fact that $Q(A) \leq 0$ is used to show either of the following:

1.
(a) \( f(A) = \mu_\{\eta = 0 \text{ on } A\} \) is superharmonic for the dual chain.
(b) \( \mu_t \{\eta = 0 \text{ on } A\} \downarrow \) in \( t \), using the semigroup property of the process.

The second uses duality as well, since duality implies that the convex cone generated by the functions \( 1_{\{\eta=0 \text{ on } A\}} \) is invariant under the semigroup.

The above theorem implies that the critical value \( \lambda_c \) grows at most like a multiple of \( \theta^{\frac{1}{2}} \) as \( \theta \uparrow \infty \). However, as proved in [4], the correct rate of growth is \( \theta^\frac{1}{2} \).

## 2 Expression for the relevant derivatives

Using the generator, and letting \( \mu_t \) be the distribution of the process at time \( t \),

\[
\frac{d}{dt} \mu_t \{\eta = 0 \text{ on } A\} = (1 + 2\theta) \sum_{x \in A} \mu_t \{\eta(x) = 1, \eta = 0 \text{ on } A \setminus \{x\}\} - \theta \sum_{x \in A, y \notin A, |x-y|=1} \mu_t \{\eta(y) = 1, \eta = 0 \text{ on } A \setminus \{x\}\} - \lambda \sum_{x \in A, y \notin A, |x-y|=1} \mu_t \{\eta(y) = 1, \eta = 0 \text{ on } A\}.
\]

Therefore, the analogue of (4.48) is

\[
Q(A) := \left. \frac{d}{dt} \mu_t \{\eta = 0 \text{ on } A\} \right|_{t=0} = (1 + 2\theta)\alpha \sum_{j \in A} L(j)R(j) - \alpha \sum_{k \notin A, k+1 \in A} L(k)[(\lambda + \theta)R(k) + \theta \rho(1)R(k+1)] - \alpha \sum_{k \in A, k+1 \notin A} [(\lambda + \theta)L(k+1) + \theta \rho(1)L(k)]R(k+1),
\]

so

\[
m_q(n) = (1 + 2\theta)\alpha \sum_{j=1}^n F(j)F(n+1-j) - 2\alpha[(\lambda + \theta)F(n+1) + \theta \rho(1)F(n)],
\]

where \( F(n) = \sum_{k=n}^\infty \rho(k) \) and \( \mu_0 \) is the renewal measure corresponding to \( \rho, F \). It follows that the analogues of (4.49) and (4.50) are

\[
\Delta m_q(1) = 2\alpha[(\lambda + \theta) - (1 + \lambda + 2\theta)\rho(1) + (\lambda + \theta)\rho(2) + \theta \rho^2(1)]
\]

and

\[
\Delta m_q(m) = \alpha \left[ (1 + 2\theta) \sum_{k,l \geq 1, k+l=m} \rho(k)\rho(l) - 2\theta \rho(1)\rho(m-1) - 2\rho(m)[1 + \lambda + 3\theta - \theta \rho(1)] + 2(\lambda + \theta)\rho(m+1) \right]
\]

for \( m > 1 \).
Proposition 2. If \( A = \{k_1, \ldots, k_n\} \) with \( k_1 < \cdots < k_n \),

\[
Q(A) = \sum_{k < l; k, l \notin A \atop k < k_n, l > k_1} L(k)R(l)\Delta q(l-k) + q(1) \sum_{j \notin A \atop k_1 < j < k_n} L(jR(j)
- (\lambda + \theta) \alpha \sum_{k \in A, k+1 \notin A} L(k + 1)[R(k) - R(k + 1)] - (\lambda + \theta) \alpha \sum_{k \notin A, k+1 \in A} [L(k + 1) - L(k)]R(k)
- 2(\lambda + \theta) \alpha \sum_{k, k+1 \notin A} [L(k + 1) - L(k)][R(k) - R(k + 1)].
\]

Proof. Letting the six displays following (4.52) of [3] be denoted by \( A_1, \ldots, A_6 \), the analogue of \( A_1 \) is

\[
\frac{1}{1 + 2\theta} \sum_{k < l; k, l \notin A \atop k < k_n, l > k_1} L(k)R(l) \left[ \alpha^{-1} \Delta q(l-k) + 2(1 + \lambda + 3\theta - \theta \rho(1))\rho(l-k) \right]
- 2(\lambda + \theta) \rho(l-k+1) \right] - \frac{2(\lambda + \theta + \theta \rho(1))}{1 + 2\theta} \sum_{k, k+1 \notin A \atop k_1 < k < k_n} L(k)R(k+1). \tag{2}
\]

This is obtained by solving the equation for \( \Delta q(m) \) above for the convolution \( \sum_{k+l=m} \rho(k)\rho(l) \) and using it to eliminate the convolution appearing in the first term on the right of (4.52).

Displays \( A_2, A_3, A_4, A_5, A_6 \) are unchanged. However, since the expression for \( \Delta q(m) \) above has a term involving \( \rho(m-1) \), they need to be supplemented with (letting \( \rho(0) = 0 \))

\[
\sum_{k < l; k, l \notin A \atop k < k_n, l > k_1} L(k)R(l)\rho(l-k-1) = \sum_{k \notin A \atop k_1 < k < k_n} L(k)R(k+1) + \sum_{l \notin A \atop k_1 < k < l} L(l)\rho(l-k-1), \tag{3}
\]

and

\[
\sum_{l \notin A \atop k < k_1} L(l)\rho(l-k-1) - \sum_{l \notin A \atop k_1 < k < l} L(l)\rho(l-k) = \sum_{l \notin A \atop l > k_1} L(l)\rho(l-k-1) = R(k_1). \tag{4}
\]

After using (2), \( A_2, A_3, A_4 \) and (3) as in the case of the contact process, one obtains an expression for \( Q(A) \) that is the sum of

\[
\sum_{k < l; k, l \notin A \atop k < k_n, l > k_1} L(k)R(l)\Delta q(l-k) + q(1) \sum_{j \notin A \atop k_1 < j < k_n} L(jR(j),
\]

terms that contain the factor \( (\lambda + \theta) \), and terms that contain the factor \( \alpha \theta \rho(1) \). The terms containing the factor \( (\lambda + \theta) \) are the same that would appear if \( \theta \) were zero, so they contribute
the remainder of the expression given in the statement of the proposition – see (4.47) in [3]. Therefore it suffices to show that the sum of the terms containing the factor $\alpha \theta \rho(1)$ vanishes. These terms are

$$
- \sum_{k \notin A, k < k_n < l} L(k) \rho(l - k) - \sum_{l \notin A, k < k_1 < l} R(l) \rho(l - k) + \sum_{k \notin A, k < k_n < l} L(k) \rho(l - k - 1) + \sum_{l \notin A, k < k_1 < l} R(l) \rho(l - k - 1)
$$

$$
-2 \sum_{k, k + 1 \notin A, k_1 < k < k_n} L(k) R(k + 1) - \sum_{k \notin A, k + 1 \in A} L(k) R(k + 1) - \sum_{k \notin A, k_1 < k < l} L(k) R(k + 1)
$$

$$
+ \sum_{l \notin A, k_1 < l < k_n} L(l - 1) R(l) + \sum_{k \notin A, k_1 < k < k_n} L(k) R(k + 1).
$$

By (4) and (5), the first four sums above contribute $L(k_n) + R(k_1)$. The $L(k_n)$ and $R(k_1)$ terms cancel the summand corresponding to $k = k_n$ in the seventh sum and the summand corresponding to $k = k_1 - 1$ in the sixth sum respectively. It follows that the sum of the terms containing the factor $\alpha \theta \rho(1)$ in $Q(A)$ vanishes.

$$
\square
$$

3 Logconvexity of the tail probabilities

By (1), $q(n) = 0$ is equivalent to $2\lambda F(2) = 1$ and

$$
(1 + 2\theta) \sum_{k=2}^{n-1} F(k) F(n+1-k) + (2 + 2\theta + \theta/\lambda) F(n) = 2(\lambda + \theta) F(n+1), \quad n \geq 2.
$$

Letting

$$
\phi(u) = \sum_{n=1}^{\infty} F(n) u^n,
$$

multiplying the above convolution equation by $u^{n+1}$ and summing, gives

$$
\lambda(1 + 2\theta) \phi^2(u) - [2\lambda(\lambda + \theta) + (2\lambda - 1)\theta u] \phi(u) + 2\lambda(\lambda + \theta) u = 0.
$$

Solving gives

$$
\phi(u) = \frac{2\lambda(\lambda + \theta) + (2\lambda - 1)\theta u - \sqrt{D(u, \lambda)}}{2\lambda(1 + 2\theta)},
$$

where

$$
D(u, \lambda) = 4\lambda^2(\lambda + \theta)^2 - 4\lambda(\lambda + \theta)(2\lambda + \theta + 2\lambda\theta)u + (2\lambda - 1)^2 \theta^2 u^2.
$$

We will consider $(\lambda, \theta)$ pairs for which $D(1, \lambda) = 0$. 

4
Proposition 3. Write
\[ 1 + u\sqrt{a} - \sqrt{(1-u)(1-au)} = \sum_{n=1}^{\infty} c_n u^n. \]

If \( a \geq 0 \), then \( c_n \geq 0 \) and \( c_n^2 \leq c_{n-1} c_{n+1} \) for \( n \geq 2 \) (i.e., \( c_n \) is logconvex). The sequence is decreasing if \( a < 1 \) and increasing if \( a > 1 \).

Proof. Here are the first few values of the sequence:
\[ c_1 = \frac{1}{2} (1 + \sqrt{a})^2, \quad c_2 = \frac{1}{8} (1 - a)^2, \quad c_3 = \frac{1}{16} (1 - a)^2 (1 + a), \quad c_4 = \frac{1}{128} (1 - a)^2 (5 + 6a + 5a^2). \]

Using generating functions, one can check that the sequence \( \{c_n, n \geq 2\} \) satisfies the recurrence
\[ 2(n+1)c_{n+1} = (2n-1)(1+a)c_n - 2(n-2)ac_{n-1} \quad (6) \]
and can be expressed as
\[ c_m = \frac{C_{m-1}}{2^{2m-1}} (1 + a^m) - \sum_{j=1}^{m-1} \frac{C_{j-1} C_{m-j-1}}{2^{2m-2}} a^j, \quad m \geq 2, \quad (7) \]
where
\[ C_n = \frac{1}{n+1} \binom{2n}{n}. \]

is the \( n \)th Catalan number. The fact that \( c_n > 0 \) follows from (7), together with the convexity of \( a^j \) as a function of \( j \) and the recursion
\[ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}. \]

To check the logconvexity, we appeal to Theorem 3.10 in [5]. Using the notation in that paper, we find that \( A = 6a(1+a), B = -12a, \) and \( C = 6(1+a) \). Therefore the present situation is covered by case (ii) of that theorem. For the final statement, note that the logconvexity of \( c_n \) implies that
\[ \frac{c_{n+1}}{c_n} \uparrow L. \]
Dividing the recursion (6) by \( nc_n \) and passing to the limit shows that \( L = 1 \) or \( L = a \). Since
\[ \frac{c_3}{c_2} = \frac{1 + a}{2}, \]
which lies between \( a \) and 1, the result follows.

\[ \begin{aligned} \text{Corollary 4. If } D(1, \lambda) &= 4\lambda^4 - 8\lambda^3 - 12\lambda^2\theta - 8\lambda\theta^2 + \theta^2 = 0 \text{ and } 4(1+\theta)\lambda^3 + 6\theta\lambda^2 + 6\theta^2\lambda - \theta^2 \geq 0, \\
&\text{then } \rho(k) \geq 0 \text{ and } F(k)^2 \leq F(k-1)F(k+1). \end{aligned} \]

Proof. Apply the proposition with
\[ a = \frac{(2\lambda - 1)^2\theta^2}{8\lambda^3(1+\theta) + 4\lambda^2\theta(3+\theta) + 8\lambda\theta^2 - \theta^2}. \]
\[ \square \]
4 Monotonicity of $L$ and $R$

In Lemma 2.12 of [1] and Proposition 4.58 of [3], this monotonicity was proved for the specific $F$ being used and under the assumption that $\rho(k)$ is logconvex respectively. Here we prove it assuming the weaker assumption of logconvexity of $F$.

**Proposition 5.** Consider the renewal measure $\mu$ corresponding to $\rho, F$. If $F$ is logconvex, then for $k_1 < \cdots < k_m$,

$$\mu\{\eta(k_1) = \cdots = \eta(k_m) = 0 \mid \eta(n) = 1\}$$

is nondecreasing in $n$ for $n > k_m$.

**Proof.** Let $u(n)$ be the corresponding renewal sequence. By breaking the complementary event up according to the leftmost $i$ for which $\eta(i) = 1$,

$$\mu\{\eta(k_1) = \cdots = \eta(k_m) = 0 \mid \eta(n) = 1\} = 1 - \sum_{j=1}^{m} u(n-k_j) \mu\{\eta(k_1) = \cdots = \eta(k_{j-1}) = 0 \mid \eta(k_j) = 1\}.$$ 

Since $u(n) \downarrow$ by Theorem 2 of [2], the result follows.

5 Limitations to this technique

One might ask whether one can do better using a different renewal measure. The answer is basically no. For the technique to work, we must have $Q(A) \leq 0$ for all $A$. Recalling that $q(n) = Q(\{1, 2, \ldots, n\})$, this would imply that the expression in (1) is nonnegative for each $n \geq 1$. Summing it on $n$ and letting $x = \sum_{n=1}^{\infty}$, gives

$$(1 + 2\theta)x^2 - 2x[\lambda + \theta + \theta \rho(1)] + 2(\lambda + \theta) \leq 0.$$ 

Using $q(1) \leq 0$, which implies that $\rho(1) \leq \frac{2\lambda - 1}{2\lambda}$, this forces the discriminant of the above quadratic to be negative if

$$\theta > 4\lambda^2 \frac{\sqrt{\lambda} - \sqrt{2}}{2\sqrt{2\lambda - 1}},$$

which is only a factor of 2 larger than expression in the statement of the theorem.

References


