Survival of the Contact+Voter Process on the Integers

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Abstract

We establish an upper bound on the critical value of the process obtained by superimposing the voter model on the contact process in one dimension.

1 Introduction

In [1], we used a rather special technique to prove that the basic contact process in one dimension survives if $\lambda = 2$. Durrett asked whether one could apply this technique to prove survival of the process obtained by adding to it a voter model interaction. This process has the following transition rates at site x:

$$0 \to 1 \text{ at rate } (\lambda + \theta)\#(y : \eta(y) = 1, |y - x| = 1),$$
$$1 \to 0 \text{ at rate } 1 + \theta\#(y : \eta(y) = 0, |y - x| = 1).$$

In this note, we work out the details of the argument in this case. Here is the result:

Theorem 1. *The process survives if*

$$\theta = \frac{4\sqrt{2}\lambda^{\frac{5}{2}} - 6\lambda^2 - 2\sqrt{2}\lambda^{\frac{3}{2}}}{8\lambda - 1} = 2\lambda^{\frac{3}{2}}\frac{\sqrt{\lambda} - \sqrt{2}}{2\sqrt{2\lambda} - 1}.$$

Note that when $\theta = 0, \lambda = 2$. The proof follows pages 168–174 of [3], using the notation there. It has three parts. They are the analogues of Proposition 4.57, Exercise 4.59 and Proposition 4.58 there. The first provides a useful expression for the time derivatives $Q(A)$ of the evolution at $t = 0$. The second gives the log-convexity of the tail probabilities of the distribution determining the initial renewal measure for the evolution, while the third uses this to prove the monotonicity that is needed to show $Q(A) \leq 0$ for all A.

Once these analogues are established, the proof of the theorem proceeds as in the case of the contact process. There are two ways of viewing the argument. The fact that $Q(A) \leq 0$ is used to show either of the following:

(a) $f(A) = \mu\{\eta = 0 \text{ on } A\}$ is superharmonic for the dual chain.

(b) $\mu_t\{\eta = 0 \text{ on } A\} \downarrow$ in t, using the semigroup property of the process.

The second uses duality as well, since duality implies that the convex cone generated by the functions $1_{\{\eta=0 \text{ on } A\}}$ is invariant under the semigroup.

The above theorem implies that the critical value λ_c grows at most like a multiple of $\theta^{\frac{2}{3}}$ as $\theta \uparrow \infty$. However, as proved in [4], the correct rate of growth is $\theta^{\frac{1}{2}}$.

2 Expression for the relevant derivatives

Using the generator, and letting μ_t be the distribution of the process at time t,

$$\begin{aligned}\frac{d}{dt}\mu_t\{\eta = 0 \text{ on } A\} &= (1+2\theta)\sum_{x\in A}\mu_t\{\eta(x) = 1, \eta = 0 \text{ on } A\backslash\{x\}\} \\ &-\theta \sum_{x\in A, y\notin A, |x-y|=1} \mu_t\{\eta(y) = 1, \eta = 0 \text{ on } A\backslash\{x\}\} \\ &-\lambda \sum_{x\in A, y\notin A, |x-y|=1} \mu_t\{\eta(y) = 1, \eta = 0 \text{ on } A\}.\end{aligned}$$

Therefore, the analogue of (4.48) is

$$\begin{aligned} Q(A) &:= \frac{d}{dt}\mu_t\{\eta = 0 \text{ on } A\}\Big|_{t=0} \\ &= (1+2\theta)\alpha\sum_{j\in A} L(j)R(j) - \alpha \sum_{k\notin A, k+1\in A} L(k)[(\lambda+\theta)R(k) + \theta\rho(1)R(k+1)] \\ &-\alpha \sum_{k\in A, k+1\notin A} [(\lambda+\theta)L(k+1) + \theta\rho(1)L(k)]R(k+1),\end{aligned}$$

so

$$q(n) = (1+2\theta)\alpha\sum_{j=1}^{n} F(j)F(n+1-j) - 2\alpha[(\lambda+\theta)F(n+1) + \theta\rho(1)F(n)], \tag{1}$$

where $F(n) = \sum_{k=n}^{\infty}\rho(k)$ and μ_0 is the renewal measure corresponding to ρ, F. It follows that the analogues of (4.49) and (4.50) are

$$\Delta q(1) = 2\alpha[(\lambda+\theta) - (1+\lambda+2\theta)\rho(1) + (\lambda+\theta)\rho(2) + \theta\rho^2(1)]$$

and

$$\begin{aligned}\Delta q(m) = \alpha\Bigg[&(1+2\theta)\sum_{k,l\geq 1; k+l=m}\rho(k)\rho(l) - 2\theta\rho(1)\rho(m-1) \\ &-2\rho(m)[1+\lambda+3\theta-\theta\rho(1)] + 2(\lambda+\theta)\rho(m+1)\Bigg]\end{aligned}$$

for $m > 1$.

Proposition 2. *If* $A=\{k_1,\dots,k_n\}$ *with* $k_1<\cdots<k_n$,

$$Q(A)=\sum_{\substack{k<l;k,l\notin A\\ k<k_n,l>k_1}} L(k)R(l)\Delta q(l-k)+q(1)\sum_{\substack{j\notin A\\ k_1<j<k_n}} L(jR(j)$$
$$-(\lambda+\theta)\alpha\sum_{k\in A,k+1\notin A} L(k+1)[R(k)-R(k+1)]-(\lambda+\theta)\alpha\sum_{k\notin A,k+1\in A}[L(k+1)-L(k)]R(k)$$
$$-2(\lambda+\theta)\alpha\sum_{k,k+1\notin A}[L(k+1)-L(k)][R(k)-R(k+1)].$$

Proof. Letting the six displays following (4.52) of [3] be denoted by $A_1,\dots,A_6$, the analogue of A_1 is

$$\begin{aligned}&\frac{1}{1+2\theta}\sum_{\substack{k<l;k,l\notin A\\ k<k_n,l>k_1}} L(k)R(l)\Big[\alpha^{-1}\Delta q(l-k)+2(1+\lambda+3\theta-\theta\rho(1))\rho(l-k)\\ &-2(\lambda+\theta)\rho(l-k+1)\Big]-2\frac{\lambda+\theta+\theta\rho(1)}{1+2\theta}\sum_{\substack{k,k+1\notin A\\ k_1<k<k_n}} L(k)R(k+1).\end{aligned}\tag{2}$$

This is obtained by solving the equation for $\Delta q(m)$ above for the convolution $\sum_{k+l=m}\rho(k)\rho(l)$ and using it to eliminate the convolution appearing in the first term on the right of (4.52).

Displays A_2,A_3,A_4,A_5,A_6 are unchanged. However, since the expression for $\Delta q(m)$ above has a term involving $\rho(m-1)$, they need to be supplemented with (letting $\rho(0)=0$)

$$\begin{aligned}\sum_{\substack{k<l;k,l\notin A\\ k<k_n,l>k_1}} L(k)R(l)\rho(l-k-1)&=\sum_{\substack{k\notin A\\ k_1<k<k_n}} L(k)R(k+1)+\sum_{\substack{l\notin A\\ k<k_1<l}} R(l)\rho(l-k-1)\\ &=\sum_{\substack{k\notin A\\ k_1<k<k_n}} L(k-1)R(k)+\sum_{\substack{k\notin A\\ k<k_n<l}} L(k)\rho(l-k-1),\end{aligned}\tag{3}$$

$$\sum_{\substack{k\notin A\\ k<k_n<l}} L(k)\rho(l-k-1)-\sum_{\substack{k\notin A\\ k<k_n<l}} L(k)\rho(l-k)=\sum_{\substack{k\notin A\\ k<k_n}} L(k)\rho(k_n-k)=L(k_n),\tag{4}$$

and

$$\sum_{\substack{l\notin A\\ k<k_1<l}} R(l)\rho(l-k-1)-\sum_{\substack{l\notin A\\ k<k_1<l}} R(l)\rho(l-k)=\sum_{\substack{l\notin A\\ l>k_1}} R(l)\rho(l-k_1)=R(k_1).\tag{5}$$

After using (2), A_2,A_3,A_4 and (3) as in the case of the contact process, one obtains an expression for $Q(A)$ that is the sum of

$$\sum_{\substack{k<l;k,l\notin A\\ k<k_n,l>k_1}} L(k)R(l)\Delta q(l-k)+q(1)\sum_{\substack{j\notin A\\ k_1<j<k_n}} L(jR(j),$$

terms that contain the factor $(\lambda+\theta)$, and terms that contain the factor $\alpha\theta\rho(1)$. The terms containing the factor $(\lambda+\theta)$ are the same that would appear if θ were zero, so they contribute the remainder of the expression given in the statement of the proposition – see (4.47) in [3]. Therefore it suffices to show that the sum of the terms containing the factor $\alpha\theta\rho(1)$ vanishes.

These terms are

$$-\sum_{\substack{k\notin A\\ k<k_n<l}} L(k)\rho(l-k)-\sum_{\substack{l\notin A\\ k<k_1<l}} R(l)\rho(l-k)+\sum_{\substack{k\notin A\\ k<k_n<l}} L(k)\rho(l-k-1)+\sum_{\substack{l\notin A\\ k<k_1<l}} R(l)\rho(l-k-1)$$
$$-2\sum_{\substack{k,k+1\notin A\\ k_1<k<k_n}} L(k)R(k+1)-\sum_{\substack{k\notin A\\ k+1\in A}} L(k)R(k+1)-\sum_{\substack{k\in A\\ k+1\notin A}} L(k)R(k+1)$$
$$+\sum_{\substack{l\notin A\\ k_1<l<k_n}} L(l-1)R(l)+\sum_{\substack{k\notin A\\ k_1<k<k_n}} L(k)R(k+1).$$

By (4) and (5), the first four sums above contribute $L(k_n)+R(k_1)$. The $L(k_n)$ and $R(k_1)$ terms cancel the summand corresponding to $k=k_n$ in the seventh sum and the summand corresponding to $k=k_1-1$ in the sixth sum respectively. It follows that the sum of the terms containing the factor $\alpha\theta\rho(1)$ in $Q(A)$ vanishes.

□

3 Logconvexity of the tail probabilities

By (1), $q(n)\equiv 0$ is equivalent to $2\lambda F(2)=1$ and

$$(1+2\theta)\sum_{k=2}^{n-1} F(k)F(n+1-k)+(2+2\theta+\theta/\lambda)F(n)=2(\lambda+\theta)F(n+1),\quad n\geq 2.$$

Letting

$$\phi(u)=\sum_{n=1}^{\infty} F(n)u^n,$$

multiplying the above convolution equation by u^{n+1} and summing, gives

$$\lambda(1+2\theta)\phi^2(u)-[2\lambda(\lambda+\theta)+(2\lambda-1)\theta u]\phi(u)+2\lambda(\lambda+\theta)u=0.$$

Solving gives

$$\phi(u)=\frac{2\lambda(\lambda+\theta)+(2\lambda-1)\theta u-\sqrt{D(u,\lambda)}}{2\lambda(1+2\theta)},$$

where

$$D(u,\lambda)=4\lambda^2(\lambda+\theta)^2-4\lambda(\lambda+\theta)(2\lambda+\theta+2\lambda\theta)u+(2\lambda-1)^2\theta^2u^2.$$

We will consider (λ,θ) pairs for which $D(1,\lambda)=0$.

Proposition 3. *Write*

$$1 + u\sqrt{a} - \sqrt{(1-u)(1-au)} = \sum_{n=1}^{\infty} c_n u^n.$$

If $a \geq 0$, then $c_n \geq 0$ and $c_n^2 \leq c_{n-1}c_{n+1}$ for $n \geq 2$ (i.e., c_n is logconvex). The sequence is decreasing if $a < 1$ and increasing if $a > 1$.

Proof. Here are the first few values of the sequence:

$$c_1 = \frac{1}{2}(1+\sqrt{a})^2, \quad c_2 = \frac{1}{8}(1-a)^2, \quad c_3 = \frac{1}{16}(1-a)^2(1+a), \quad c_4 = \frac{1}{128}(1-a)^2(5+6a+5a^2).$$

Using generating functions, one can check that the sequence $\{c_n, n \geq 2\}$ satisfies the recurrence

$$2(n+1)c_{n+1} = (2n-1)(1+a)c_n - 2(n-2)ac_{n-1} \tag{6}$$

and can be expressed as

$$c_m = \frac{C_{m-1}}{2^{2m-1}}(1+a^m) - \sum_{j=1}^{m-1} \frac{C_{j-1}C_{m-j-1}}{2^{2m-2}} a^j, \quad m \geq 2, \tag{7}$$

where

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

is the nth Catalan number. The fact that $c_n > 0$ follows from (7), together with the convexity of a^j as a function of j and the recursion

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

To check the logconvexity, we appeal to Theorem 3.10 in [5]. Using the notation in that paper, we find that $A = 6a(1+a), B = -12a$, and $C = 6(1+a)$. Therefore the present situation is covered by case (ii) of that theorem. For the final statement, note that the logconvexity of c_n implies that

$$\frac{c_{n+1}}{c_n} \uparrow L.$$

Dividing the recursion (6) by nc_n and passing to the limit shows that $L = 1$ or $L = a$. Since

$$\frac{c_3}{c_2} = \frac{1+a}{2},$$

which lies between a and 1, the result follows. □

Corollary 4. *If $D(1,\lambda) = 4\lambda^4 - 8\lambda^3 - 12\lambda^2\theta - 8\lambda\theta^2 + \theta^2 = 0$ and $4(1+\theta)\lambda^3 + 6\theta\lambda^2 + 6\theta^2\lambda - \theta^2 \geq 0$, then $\rho(k) \geq 0$ and $F(k)^2 \leq F(k-1)F(k+1)$.*

Proof. Apply the proposition with

$$a = \frac{(2\lambda-1)^2\theta^2}{8\lambda^3(1+\theta) + 4\lambda^2\theta(3+\theta) + 8\lambda\theta^2 - \theta^2}.$$

□

4 Monotonicity of L and R

In Lemma 2.12 of [1] and Proposition 4.58 of [3], this monotonicity was proved for the specific F being used and under the assumption that $\rho(k)$ is logconvex respectively. Here we prove it assuming the weaker assumption of logconvexity of F.

Proposition 5. *Consider the renewal measure μ corresponding to ρ, F. If F is logconvex, then for $k_1 < \cdots < k_m$,*

$$\mu\{\eta(k_1) = \cdots = \eta(k_m) = 0 \mid \eta(n) = 1\}$$

is nondecreasing in n for $n > k_m$.

Proof. Let $u(n)$ be the corresponding renewal sequence. By breaking the complementary event up according to the leftmost i for which $\eta(i) = 1$,

$$\mu\{\eta(k_1) = \cdots = \eta(k_m) = 0 \mid \eta(n) = 1\} = 1 - \sum_{j=1}^{m} u(n-k_j)\mu\{\eta(k_1) = \cdots = \eta(k_{j-1}) = 0 \mid \eta(k_j) = 1\}.$$

Since $u(n) \downarrow$ by Theorem 2 of [2], the result follows. □

5 Limitations to this technique

One might ask whether one can do better using a different renewal measure. The answer is basically no. For the technique to work, we must have $Q(A) \leq 0$ for all A. Recalling that $q(n) = Q(\{1, 2, \dots, n\})$, this would imply that the expression in (1) is nonnegative for each $n \geq 1$. Summing it on n and letting $x = \sum_{n=1}^{\infty}$, gives

$$(1 + 2\theta)x^2 - 2x[\lambda + \theta + \theta\rho(1)] + 2(\lambda + \theta) \leq 0.$$

Using $q(1) \leq 0$, which implies that $\rho(1) \leq \frac{2\lambda-1}{2\lambda}$, this forces the discriminant of the above quadratic to be negative if

$$\theta > 4\lambda^{\frac{3}{2}} \frac{\sqrt{\lambda} - \sqrt{2}}{2\sqrt{2\lambda} - 1},$$

which is only a factor of 2 larger than expression in the statement of the theorem.

References

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