The Asymptotic Shapley Value for a Simple Market Game

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Abstract

We consider the game in which b buyers each seek to purchase 1 unit of an indivisible good from s sellers, each of whom has k units to sell. The good is worth 0 to each seller and 1 to each buyer. Using results from Brownian motion, we find a closed form solution for the limiting Shapley value as s and b increase without bound. This asymptotic value depends upon the seller size k, the limiting ratio b/ks of buyers to items for sale, and the limiting ratio $[ks - b]/\sqrt{b+s}$ of the excess supply relative to the square root of the number of market participants.

1 Introduction

In this note we consider a simple market game with many symmetric buyers and sellers and obtain a new asymptotic result regarding the Shapley value of this game. The approach used to establish our result is related to classical limit theorems on convergence to tied down Brownian motion (the Brownian bridge).

Shapley and Shubik [5] initiated the program of comparing the core, competitive outcome, and Shapley values in symmetric games with many players. Since then, it has been established that the Shapley value of a game and the competitive outcome are identical with a continuum of non-atomic players ([1]). However, in an atomistic game this equivalence need not hold. This is easily shown by the simple market game described in the next paragraph.

In our simple market game each of s sellers possesses k units of an indivisible good to sell, and the good is worth 0 to each seller. There are b buyers each seeking to purchase 1 unit, and the good is worth 1 to each buyer. (Here k, s, and b are positive integers.) This is the simplest game that captures trade in a market with many agents. When $k > 1$, each seller is a "store" with k units for sale.

Suppose that in this market game there are 1 million sellers, 1 millon plus 1 buyers, and $k = 1$ so each seller has one object to sell. The core and the competitive equilibrium of this game entail a price of 1 because demand for the good exceeds the amount available for sale. If we add just two more sellers, the competitive price drops from 1 to 0. By contrast, the Shapley value is near 1/2 in both cases.

This example exhibits a sharp discontinuity in the competitive equilibrium of this atomistic market game. It also suggests that the Shapley value presents a smooth equilibrium without any such discontinuity. Traditionally, this smoothing effect has been viewed as the natural result of the Shapley value's origins in cooperative game theory, involving the balancing of probabilities of various coalitions being formed. However, recently there has been considerable interest in non-cooperative rationales for the Shapley value. Two influential examples are Gul's [2] model of random bilaterial contracting and Hart and Mas-Colell's [3] model of sequential bilateral offers.

While the behavior of the Shapley value is clear when there are a small numbers of players, its behavior is neither well understood nor, in general, easy to compute as the number of players grows very large. Shapley and Shubik [5] analyzed this problem with $k = 1$ for two special cases and provided asymptotic results. In the first case the ratio b/s of buyers to sellers is fixed at $b/s = \alpha > 1$: as the number of players increases, the Shapley value converges to 1 and coincides with the core. When $b/s = \alpha < 1$, the Shapley value converges to 0, as does the core. In the second case they analyzed, the excess of buyers over sellers is fixed so $b/s \to \alpha = 1$: the Shapley value converges to 1/2 regardless of the value of the fixed excess. The core produces a price of 1 when the fixed excess is positive and a price of 0 when it is negative. With a fixed excess of 0, the core contains all prices in the range $[0, 1]$.

In this note we show that the asymptotic Shapley value for this market game is not restricted to the values 0, 1/2, and 1; instead, all values between 0 and 1 are possible. Let $M \equiv b+s$ denote the size of the market, and let $d \equiv ks - b$ denote the excess supply. We provide a closed form expression for the limit of the Shapley value of this game as a function of $u \equiv \lim_{b,s\to\infty} d/\sqrt{M}$. As long as the difference d between units for sale and buyers grows in proportion to the square root of the size of the market, the asymptotic Shapley value (for each unit) can take on any intermediate value between 0 and 1.

2 The Closed Form Solution

We now present a closed form solution that gives the asymptotic Shapley value as s and b increase to ∞. A particularly interesting aspect of our solution is that for each price p between 0 and 1, there are growth rates for s and b such that the Shapley value converges to p. This stands in stark contrast to the other solution concepts which yield prices of 0, 1, or an indeterminate price (or all prices) between 0 and 1.

To begin, set $k = 1$, where k is the number of units each seller possesses, and let $V(b, s)$ denote the Shapley value for a seller in this game (naturally, the value added by all players is $v \equiv \min(b, s)$ so that each buyer's Shapley value is $[v - sV(b, s)]/b)$. To compute $V(b, s)$, consider those permutations of the $b + s$ players in which a given seller is the $i + 1^{st}$ player. There is some number j of buyers amongst the first i players in the permutation whence there are $i - j$ sellers amongst the first i players. This seller's value added is 1 if the number j of buyers is strictly greater than the number $i - j$ of sellers: the seller's value added is 1 if and only if $2j > i$. Clearly, the probability that this seller is the $i + 1^{st}$ player is $1/(b + s)$. The product $\binom{b}{j}\binom{s-1}{i-j}/\binom{s+b-1}{i}$ of binomial coefficients is the probability that the first i players in a permutation has j buyers and $i-j$ sellers given that the $i+1^{st}$ player is a seller. Consequently, $V(b, s)$ can be written as

$$V(b, s) = \frac{1}{b+s} \sum_{i=0}^{s+b-1} \sum_{2j>i} \frac{\binom{b}{j}\binom{s-1}{i-j}}{\binom{s+b-1}{i}}.$$

When the ratio b/ks of buyers b to objects ks available for sale converges to 1, the limit of the Shapley value $V(b, s)$ converges to the expression with the integral given given in (1). This is not intuitive. Nevertheless, upon reflection, it is unsurprising that the asymptotic Shapley value is closely related to tied down Brownian motion. To see the connection between $V(b,s)$ and tied down Brownian motion, fix $k=1$, let b and s be given, define $M=b+s$, and suppose that b/s converges to α. For each of the $\binom{M}{s}$ permutations of the M players, think of the player in the j^{th} position as having arrived at time j/M. Define the random variable X_j to be 1 when the player in the j^{th} position is a buyer and -1 when the player in the j^{th} position is a seller so that in the limit $E(X_j)=(\alpha-1)/(\alpha+1)$. The Shapley value is the fraction of time that an arriving seller finds $S_n=\sum_{j=1}^{M} X_j$ positive. If the X_j were independent, then $\{S_n : n=0,1,\dots,M\}$ appropriately scaled would (by Donsker's Theorem) converge to Brownian motion. But for each and every permutation of the M players $S_M=b-s$ and, of course, the X_j are not independent. Instead, $\{S_n : n=0,1,\dots,M\}$ converges to tied down Brownian motion.

More generally, when each seller is a store of integer size k with $k\ge 1$, a similar argument to the one above reveals that the expression for $V(s,b)$, the Shapley value for each seller (who has k units for sale), is

$$V(b,s)=\frac{1}{b+s}\sum_{i=0}^{s+b-1}\sum_{(k+1)j>ik}\min\{k,j-(i-j)k\}\frac{\binom{b}{j}\binom{s-1}{i-j}}{\binom{s+b-1}{i}}.$$

Theorem: Suppose $b,s\to\infty$ so that $b/ks\to\alpha$.

(a) If $\alpha<1$, then $V(b,s)\to 0$.

(b) If $\alpha>1$, then $V(b,s)\to k$.

(c) If $\alpha=1$, suppose that

$$\frac{ks-b}{\sqrt{b+s}}\to u.$$

Then

$$V(b,s)\to \frac{k^2}{\sqrt{2\pi}}\int_0^\infty \frac{x^2}{u^2+kx^2}e^{-x^2/2}dx \quad \text{if u} \ge 0, \tag{1a}$$

and

$$V(b,s)\to k-\frac{k^2}{\sqrt{2\pi}}\int_0^\infty \frac{x^2}{u^2+kx^2}e^{-x^2/2}dx \quad \text{if u} \le 0. \tag{1b}$$

Proof : We first write V in terms of a simple random walk. Take $0<p<1$, let $X_1,X_2,...$ be independent identically distributed Bernoulli random variables with

$$P(X_i=1)=p,\quad P(X_i=0)=q=1-p,$$

and let $S_m = X_1 + \cdots + X_m$ be the corresponding partial sums. Setting $N = b + s - 1$, we see that

$$P(S_b = j \mid S_N = i) = \frac{P(S_b = j)P(S_{s-1} = i - j)}{P(S_N = i)} = \frac{\binom{b}{j}p^j q^{b-j}\binom{s-1}{i-j}p^{i-j}q^{s-1-i+j}}{\binom{N}{i}p^i q^{N-i}} = \frac{\binom{b}{j}\binom{s-1}{i-j}}{\binom{N}{i}}.$$

Therefore,

$$\begin{aligned}
V(b,s) \quad &= \frac{1}{b+s}\sum_{i=0}^{s+b-1}\sum_{(k+1)j>ik}\min\{k, j-(i-j)k\}P(S_b = j \mid S_N = i) \\
&= \frac{1}{N+1}\sum_{i=0}^{N} E\Big\{\min\{k,(k+1)S_b - ik\}; (k+1)S_b > ik \mid S_N = i\Big\}, \\
&= \int_0^1 E\Big\{\min\{k, Y(b,s,p)\}; Y(b,s,p) > 0 \mid S_N = \lfloor (N+1)p \rfloor\Big\} dp, \qquad (2)
\end{aligned}$$

where $Y(b,s,p) = (k+1)S_b - k\lfloor (b+s)p \rfloor$. Here $\lfloor a \rfloor$ is the greatest integer $\leq a$, and $E(X;A)$ means $E(X1_A)$ if X is a random variable and A is an event. Note that $Y(b,s,p)$ depends on p in two ways – explicitly in the second term, and implicitly through the distribution of the X_i's in the first.

The central limit theorem asserts that

$$\frac{S_b - bp}{\sqrt{Npq}} \Rightarrow \mathcal{N}\left(0, \frac{k\alpha}{k\alpha + 1}\right),$$

where $\Rightarrow$ denotes convergence in distribution and $\mathcal{N}(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2. A consequence of this is that the following conditional distribution is also asymptotically normal, but with a smaller variance:

$$\left(\frac{S_b - bp}{\sqrt{Npq}}\middle| S_N = \lfloor (N+1)p \rfloor\right) \Rightarrow \mathcal{N}\left(0, \frac{k\alpha}{(k\alpha+1)^2}\right). \qquad (3)$$

This is a special case of Theorem 4 of [4], but can be checked directly by writing

$$P(S_b = l \mid S_N = m) = P(S_b = l)\frac{P(S_{N-b} = m - l)}{P(S_N = m)},$$

and applying Stirling's formula to the binomial coefficients that appear in the second factor above for $m = \lfloor (N+1)p \rfloor$ and $l = bp + y\sqrt{Npq}$. Specifically, one obtains

$$\frac{P(S_{N-b} = m - l)}{P(S_N = m)} \to \sqrt{k\alpha + 1}\, e^{-y^2(k\alpha+1)/2}.$$

This factor converts one normal density into the other.

By (3), the integrand in (2) tends to 0 if $\alpha < 1$, to k if $\alpha > 1$, and to

$$kP(Z \geq u\sqrt{p/(qk)})$$

if $\alpha = 1$, where Z denotes a standard normal random variable. By the bounded convergence theorem, this gives parts (a) and (b) of the theorem. If $\alpha = 1$, it gives

$$V(b,s) \to k \int_0^1 P(Z \geq u\sqrt{p/(qk)})dp.$$

Now take $u \geq 0$ (the case $u \leq 0$ is similar). An interchange of the order of integration gives

$$\int_0^1 P(Z \geq u\sqrt{p/(qk)})dp = \frac{1}{\sqrt{2\pi}} \int_0^1 \int_{u\sqrt{p/(qk)}}^{\infty} e^{-x^2/2} dx dp = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{kx^2}{u^2 + kx^2} e^{-x^2/2} dx$$

as required. ■

Note that $V(b,s)$ is the value per seller. The equilibrium market price for each unit of the item is $V(b,s)/k$. In a large market, the closed form solution reveals that when $0 < u < \infty$, $\lim_{b,s\to\infty} V(b,s)/k$ is strictly increasing in k, with limit $1/2$: when there is an excess supply ($ks > b$), larger stores induce an increase in the equilibrium market price. (The opposite is true when $-\infty < u < 0$.)

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