T. Liggett Mathematics 31B/4 – Midterm Solutions November 17, 2004

(10) 1. In each case, state whether the integral converges or diverges. No explanation is needed.

(a) $\displaystyle\int_1^\infty \frac{\cos^2 x}{(3x+1)^2}dx$ Converges

(b) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ Converges

(c) $\displaystyle\int_1^\infty \frac{1}{x}dx$ Diverges

(d) $\displaystyle\int_{-\infty}^\infty e^{-|x|}dx$ Converges

(e) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}(1+x)}dx$ Converges

(15) 2. (a) Use logarithmic differentiation to compute y' where

$$y = (\ln x)^{\cos x}$$

Solution: $\ln y = \cos x \ln \ln x$, so that

$$\frac{y'}{y} = -\sin x \ln \ln x + \frac{\cos x}{x \ln x}.$$

Therefore,

$$y' = (\ln x)^{\cos x}\Big(-\sin x \ln \ln x + \frac{\cos x}{x \ln x}\Big).$$

(b) Find

$$\frac{d^9}{dx^9}(x^8 \ln x)$$

Solution:

$$\frac{d^9}{dx^9}(x^8 \ln x) = \frac{d^8}{dx^8}(x^7 + 8x^7 \ln x) = 8\frac{d^8}{dx^8}(x^7 \ln x),$$

since the eighth derivative of x^7 is zero. Repeating this a number of times gives

$$\frac{d^9}{dx^9}(x^8 \ln x) = \frac{8!}{x}.$$

(15) 3. (a) Evaluate

$$\lim_{x\to\infty} \arccos\left(\frac{1+x^2}{1+2x^2}\right)$$

Solution:

$$\lim_{x\to\infty} \arccos\left(\frac{1+x^2}{1+2x^2}\right) = \arccos(1/2) = \frac{\pi}{3}.$$

(b) Differentiate to show that

$$\sin^{-1} x + \cos^{-1} x$$

is a constant. What is the value of the constant?
Solution:

$$\frac{d}{dx}\left(\sin^{-1} x + \cos^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0.$$

Therefore, $\sin^{-1} x + \cos^{-1} x$ is constant. The constant is $\sin^{-1} 0 + \cos^{-1} 0 = \pi/2$.

(15) 4. Integrate:

(a)

$$\int (\sin x)^3 (\cos x)^2 dx$$

Solution: Let $u = \cos x, du = -\sin x dx$. Then

$$\int (\sin x)^3 (\cos x)^2 dx = -\int u^2(1-u^2)du = -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.$$

(b)

$$\int x\sqrt{1-x^4}dx$$

Solution: Let $u = x^2, du = 2xdx$, and then $u = \sin v, du = \cos v dv$:

$$\int x\sqrt{1-x^4}dx = \frac{1}{2}\int \sqrt{1-u^2}du = \frac{1}{2}\int \cos^2 v dv = \frac{1}{4}\int (1+\cos(2v))dv = \frac{v}{4} + \frac{1}{8}\sin(2v) + C.$$

Therefore,

$$\int x\sqrt{1-x^4}dx = \frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}\sin v\cos v + C = \frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}x^2\sqrt{1-x^4}.$$

(15) 5. Integrate

$$\int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4}dx$$

Solution:
$$\frac{x^3-2x^2+x+1}{x^4+5x^2+4} = \frac{a+bx}{x^2+1} + \frac{c+dx}{x^2+4}.$$
Crossmultiplying gives
$$x^3-2x^2+x+1 = (a+bx)(x^2+4)+(c+dx)(x^2+1).$$
Equating coefficients gives $1 = b+d, -2 = a+c, 1 = 4b+d, 1 = 4a+c$. Solving gives $a = 1, b = 0, c = -3, d = 1$. Therefore,
$$\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}dx = \int \left(\frac{1}{x^2+1}+\frac{x-3}{x^2+4}\right)dx = \tan^{-1} x+\frac{1}{2}\ln(x^2+4)-\frac{3}{2}\tan^{-1}\frac{x}{2}+C.$$
(15) 6. (a) Evaluate
$$\lim_{x\to\infty} \frac{(\ln x)^2}{x}$$
Solution: By L'Hospital's rule,
$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2}{x} = 0.$$
(b) For what values of a and b is the following statement true?
$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}\right) = 0$$
Solution:
$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}\right) = a + \lim_{x\to 0}\frac{\sin 2x + bx}{x^3}.$$
By L'Hospital's rule, $b = -2$, since otherwise the limit is $\pm\infty$. With this choice,
$$\lim_{x\to 0}\frac{\sin 2x - 2x}{x^3} = \lim_{x\to 0}\frac{2\cos 2x - 2}{3x^2} = \lim_{x\to 0}\frac{-4\sin 2x}{6x} = -\frac{4}{3}.$$
Therefore, $a = 4/3$.

(15) 7. Find the length of the curve
$$y = \frac{x^2}{2} - \frac{\ln x}{4}, \quad 2 \le x \le 4.$$
Solution:
$$L = \int_2^4 \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx = \int_2^4 \left(x+\frac{1}{4x}\right)dx = 6+\frac{1}{4}\ln 2.$$