Mathematics 31B/4 – Fall 2004 – Quiz Solutions

Quiz for Tuesday, December 7

1. Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Explain your answer briefly.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$

Solution: Converges conditionally. It converges by the alternating series test, but does not converge absolutely. To see that it does not converge absolutely, compare the series without the alternating signs with the harmonic series:

$$\frac{n}{n^2+1} \geq \frac{1}{2n}.$$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{3-\cos n}{n^{2/3}-2}$

Solution: Diverges, by the comparison test:

$$\frac{3-\cos n}{n^{2/3}-2} \geq \frac{2}{n^{2/3}}.$$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{2^n n!}{5\cdot 8\cdot 11\cdots(3n+2)}$

Solution: Converges absolutely, by the ratio test: Letting

$$a_n = \frac{2^n n!}{5\cdot 8\cdot 11\cdots(3n+2)},$$

we see that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2(n+1)}{3n+5} = \frac{2}{3} < 1.$$

2. Find the radius of convergence and the interval of convergence of the series.

(a) $\displaystyle\sum_{n=0}^{\infty} n^3(x-5)^n$

Solution: For the radius of convergence, use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}\right| = |x-5|.$$

Therefore, the series converges absolutely if $|x-5|<1$ and diverges if $|x-5|>1$. So, $R=1$. To test the endpoints, note that if $x=4$ or $x=6$, then the series diverges, since the summands do not tend to zero. So, the interval of convergence is $(4,6)$.

(b) $$\sum_{n=1}^{\infty} \frac{n(x-4)^n}{n^3+1}$$

Solution: Again, $R=1$, but now, when $x=3$ or $x=5$, the series converges by comparison with $\sum n^{-2}$.

Quiz for Thursday, December 9

1. Find the radius of convergence and the interval of convergence of the series.

(a) $$\sum_{n=0}^{\infty} \sqrt{n} x^n$$

Solution: Use the ratio test:

$$\lim_{n\to\infty} \left| \frac{\sqrt{n+1} x^{n+1}}{\sqrt{n} x^n} \right| = |x|,$$

so $R=1$. When $x=\pm 1$, the summands do not tend to zero, so the series diverges. Therefore, the interval of convergence is $(-1,1)$.

(b) $$\sum_{n=2}^{\infty} (-1)^n \frac{(2x+3)^n}{n \ln n}$$

Solution: Use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(2x+3)^{n+1} n \ln n}{(2x+3)^n (n+1) \ln(n+1)} \right| = |2x+3|.$$

Therefore $R=1/2$. The series converges at $x=-1$ (by the alternating series test) and diverges at $x=-2$ (by the integral test), so the interval of convergence is $(-2,-1]$.

2. Let

$$f(x) = \frac{x^2}{(1-2x)^2}.$$

(a) Find a power series representation for f, and determine its radius of convergence.
Solution: Start with the geometric series

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n,$$

and differentiate

$$\frac{2}{(1-2x)^2} = \sum_{n=0}^{\infty} n 2^n x^{n-1}.$$

Now multiply by $x^2/2$:

$$\frac{x^2}{(1-2x)^2} = \sum_{n=0}^{\infty} n2^{n-1}x^{n+1}.$$

By the ratio test, $R = 1/2$.
(b) Find a power series representation for $\int f(x)dx$.
Solution: Integrate term by term:

$$\int \frac{x^2}{(1-2x)^2}dx = \sum_{n=0}^{\infty} \frac{n}{n+2}2^{n-1}x^{n+2} + C.$$