Mathematics 31B/4 – Fall 2004 – Quiz Solutions

Quiz for Thursday, November 18

1. For a sequence a_n, define:

(a) a_n is bounded below.

Solution: There is an A so that $a_n \geq A$ for all n.

(b) $\lim_{n\to\infty} a_n = L$.

Solution: For every $\epsilon > 0$ there is an N so that $|a_n - L| < \epsilon$ for all $n > N$.

2. In each case, find $\lim_{n\to\infty} a_n$ if it exists; otherwise say it does not exist:

(a) $a_n = (-1)^n$

Solution: Does not exist.

(b) $a_n = \frac{n^2+1}{n+1}$

Solution: Does not exist.

(c) $a_n = \frac{\sin n}{n}$

Solution: Limit is 0.

(d) $a_n = \frac{6n^2+5}{2n^2-3n}$

Solution: Limit is 3.

(e) $a_n = \sqrt{n+3} - \sqrt{n}$

Solution: Limit is 0, since

$$a_n = \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+3} + \sqrt{n})}{(\sqrt{n+3} + \sqrt{n})} = \frac{3}{(\sqrt{n+3} + \sqrt{n})}.$$

Quiz for Tuesday, November 23

1. In each case, determine whether the series converges or diverges. If it converges, find the sum of the series.

(a) $$\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$$

Solution: Converges;

$$\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = e \sum_{n=1}^{\infty} \frac{e^{n-1}}{3^{n-1}} = \frac{e}{1-(e/3)} = \frac{3e}{3-e}.$$

(b) $$\sum_{n=1}^{\infty} \frac{3}{n}$$

Solution: Diverges

(c) $$\sum_{n=1}^{\infty} \frac{2}{(n+3)(n+1)}$$

Solution: Converges;

$$\sum_{n=1}^{\infty} \frac{2}{(n+3)(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+3} \right)$$
$$= \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{N+1} - \frac{1}{N+3} \right)$$
$$= \lim_{N\to\infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{N+2} - \frac{1}{N+3} \right) = \frac{5}{6}.$$

(d) $$\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5} \right)$$

Solution: Diverges, since the summands do not tend to 0.
2. (a) Define: $\sum_{n=1}^{\infty} a_n$ converges.
Solution: $s = \lim_{N\to\infty} s_N$ exists, where $s_N = \sum_{n=1}^{N} a_n$.
(b) Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0.$$

Quiz for Tuesday, November 30

1. (a) Show that $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.
Solution: Use the integral test;

$$\int_1^{\infty} xe^{-x^2}\,dx$$

converges. To check the required monotonicity, write

$$\frac{d}{dx} xe^{-x^2} = (1-2x^2)e^{-x^2} \le 0, \quad x \ge 1.$$

(b) Find good upper and lower bounds for $\sum_{n=4}^{\infty} ne^{-n^2}$. (Note that the sum starts at $n = 4$!)
Solution:

$$\frac{1}{2}e^{-16} = \int_4^{\infty} xe^{-x^2}\,dx \le \sum_{n=4}^{\infty} ne^{-n^2} \le \int_3^{\infty} xe^{-x^2}\,dx = \frac{1}{2}e^{-9}.$$

2. Determine whether the following series converge or diverge. Explain your answers.

(a) $$\sum_{n=1}^{\infty} \frac{2}{n^3+4}$$

Solution: Converges, by comparison with $\sum 2/n^3$.

(b) $$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$$

Solution: Diverges, by the limit comparison test, applied to the harmonic series.

(c) $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Solution: Converges, by comparison with $\sum_{n=1}^{\infty} 2/n^2$:

$$\frac{n!}{n^n} = \frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{2}{n}\frac{1}{n} \leq \frac{2}{n^2}.$$

Quiz for Thursday, December 2

1. (a) Show that the series $\sum_{n=1}^{\infty}(-1)^n/n^3$ converges.
Solution: Apply the alternating series test.
(b) How many terms of the series $\sum_{n=1}^{\infty}(-1)^n/n^3$ do we need to add to find the sum to within an error of less than .001?
Solution: Using $|s - s_n| \leq 1/(n+1)^3$, we see that we need $1/(n+1)^3 \leq .001$, or $n \geq 9$.
2. Determine whether the following series converge or diverge. Explain your answers.

(a) $$\sum_{n=1}^{\infty} \frac{n-1}{n^2}$$

Solution: Diverges, by applying the limit comparison test to the harmonic series.

(b) $$\sum_{n=1}^{\infty} \frac{n^2+1}{3n^4-1}$$

Solution: Converges, by applying the limit comparison test to $\sum 1/n^2$.

(c) $$\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$$

Solution: Converges, by applying the limit comparison test to the geometric series $\sum(2/3)^n$.