Mathematics 31B/4 – Fall 2004 – Quiz Solutions

Quiz for Tuesday, November 2

1. Evaluate

(a)
$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

Solution: This is an indeterminate form, so we may apply L'Hospital's rule, to get

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x}$$

if the latter limit exists. This again is indeterminate, so we may apply L'Hospital's rule again to get

$$\lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.$$

(b)
$$\lim_{x \to 0} (1 - 2x)^{1/x}$$

Solution: Taking logs, we see that we must evaluate

$$\lim_{x \to 0} \frac{\ln (1 - 2x)}{x}.$$

This is an indeterminate form, so

$$\lim_{x \to 0} \frac{\ln (1 - 2x)}{x} = \lim_{x \to 0} \frac{-2}{1 - 2x} = -2.$$

Therefore

$$\lim_{x \to 0} (1 - 2x)^{1/x} = e^{-2}.$$

2. Evaluate

$$\int_1^2 x^4 (\ln x)^2 dx$$

Solution: Use integration by parts twice.

$$\begin{aligned}
\int_1^2 x^4 (\ln x)^2 dx &= \frac{x^5}{5} (\ln x)^2 \Big|_1^2 - \frac{2}{5} \int_1^2 x^4 \ln x dx \\
&= \frac{32}{5} (\ln 2)^2 - \frac{2}{25} x^5 \ln x \Big|_1^2 + \frac{2}{25} \int_1^2 x^4 dx \\
&= \frac{32}{5} (\ln 2)^2 - \frac{64}{25} \ln 2 + \frac{62}{125}.
\end{aligned}$$

Quiz for Thursday, November 4

1. Evaluate

(a)
$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

Solution: This is an indeterminate form, so we may apply L'Hospital's rule, to get

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2}$$

if the latter limit exists. This again is indeterminate, so we may apply L'Hospital's rule again to get

$$\lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin x}{6x} = -\frac{1}{6}.$$

(b)

$$\lim_{x \to \infty} \left(1 - \frac{a}{x}\right)^{bx}$$

Solution: Taking logs, we see that we must evaluate

$$\lim_{x \to \infty} bx \ln\left(1 - \frac{a}{x}\right) = b \lim_{x \to \infty} \ln\left(1 - \frac{a}{x}\right) \Big/ (1/x).$$

By L'Hospital, this limit is

$$b \lim_{x \to \infty} \frac{ax^{-2}}{-x^{-2}} = -ab.$$

Therefore

$$\lim_{x \to \infty} \left(1 - \frac{a}{x}\right)^{bx} = e^{-ab}.$$

2. Evaluate

$$\int \cos x \ln(\sin x) dx$$

Solution: Integrating by parts, we have

$$\int \cos x \ln(\sin x) dx = \sin x \ln(\sin x) - \int \cos x dx = \sin x \ln(\sin x) - \sin x + C.$$

Quiz for Thursday, November 9

1. Evaluate

$$\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)} dx$$

Solution: Write

$$\frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2+1}.$$

Crossmultiplying gives

$$x^2 - 2x - 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2.$$

Setting $x = 1$ yields $B = -1$, and using this,

$$2x(x-1) = A(x-1)(x^2+1) + (Cx+D)(x-1)^2.$$

Divide by $(x-1)$ and then let $x = 1$ to get $A = 1$, and using this,

$$0 = (x-1)^2 + (Cx+D)(x-1).$$

Divide by $(x-1)$ again to conclude that $C = -1$ and $D = 1$. Therefore,

$$\begin{aligned}\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx &= \int \left(\frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1-x}{x^2+1}\right)dx \\ &= \ln(x-1) + \frac{1}{x-1} + \arctan x - \frac{1}{2}\ln(x^2+1).\end{aligned}$$

2. Determine whether the following integrals converge or diverge. Be sure to explain why this is the case. For those that converge, find the value.

(a)
$$\int_1^9 \frac{1}{\sqrt[3]{x-9}}dx$$

Solution: We need to consider the existence of

$$\lim_{a\to 9}\int_1^a \frac{1}{\sqrt[3]{x-9}}dx = \lim_{a\to 9}\left(\frac{3}{2}(a-9)^{2/3} - 6\right) = -6.$$

Therefore, the integral converges and has value -6.

(b)
$$\int_0^2 \frac{x-3}{2x-3}dx$$

Solution: This time,

$$\lim_{a\to\frac{3}{2}^-}\int_0^a \frac{x-3}{2x-3}dx = \lim_{a\to\frac{3}{2}^-}\int_0^a \left[\frac{1}{2} - \frac{3}{2}\frac{1}{2x-3}\right]dx = \frac{3}{4} - \frac{3}{4}\lim_{a\to\frac{3}{2}^-}\ln(3-2x)\Big|_0^a = +\infty,$$

so the integral diverges.

(c)
$$\int_e^\infty \frac{1}{x\ln x}dx$$

Solution: Let $u = \ln x$:

$$\lim_{a\to\infty}\int_e^a \frac{1}{x\ln x}dx = \lim_{a\to\infty}\int_1^{\ln a}\frac{1}{u}du = \lim_{a\to\infty}\ln(\ln a) = \infty,$$

so the integral diverges.