(15) 1. Define the following:

(a) The series $\sum_n a_n$ converges.

Solution: Let $s_n = \sum_{k=1}^n a_k$ be the nth partial sum of the series. Then $\sum_n a_n$ converges if $\lim_{n\to\infty} s_n$ exists and is finite.

(b) The series $\sum_n a_n$ converges absolutely.

Solution: $\sum_n a_n$ converges absolutely if $\sum_n |a_n|$ converges.

(c) The series $\sum_n a_n$ converges conditionally.

Solution: $\sum_n a_n$ converges conditionally if it converges, but does not converge absolutely.

(10) 2. State the integral test for convergence of series.

Solution: Suppose f is a positive, continuous, decreasing function on $[1,\infty)$, and let $a_n = f(n)$. Then $\sum a_n$ converges iff $\int_1^\infty f(x)dx$ converges.

(10) 3. Compute

(a)
$$\lim_{x\to 2} \frac{\sqrt{2x}-2}{\log(x-1)}$$

Solution: This is a 0/0 form, so by L'Hospital's rule,

$$\lim_{x\to 2} \frac{\sqrt{2x}-2}{\log(x-1)} = \lim_{x\to 2} \frac{x-1}{\sqrt{2x}} = \frac{1}{2}.$$

(b)
$$\lim_{x\to 1} \frac{x^4-4x^3+8x-5}{x^3-3x+2}$$

Solution: Again by L'Hospital (applied twice),

$$\lim_{x\to 1} \frac{x^4-4x^3+8x-5}{x^3-3x+2} = \lim_{x\to 1} \frac{4x^3-12x^2+8}{3x^2-3} = \lim_{x\to 1} \frac{12x^2-24x}{6x} = -2.$$

(20) 4. For each of the series below, determine whether it converges absolutely, converges conditionally, or diverges. Explain briefly.

(a)
$$\sum_{n=2}^{\infty} \frac{(-1)^n\sqrt{n}}{\log n}$$

Solution: Diverges, since the summands do not tend to zero.

(b)
$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$$

Solution: Converges absolutely by the comparison test, since

$$\frac{n!}{n^n} \le \frac{2}{n^2}.$$

(c)
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}\log n}{n^2+2}$$

Solution: Converges absolutely, by the comparison test. Since $\log n/n^{1/4}$ tends to zero, it is bounded by some constant C. Therefore,

$$\frac{\sqrt{n}\log n}{n^2+2} \leq \frac{Cn^{3/4}}{n^2+2} \leq C\frac{1}{n^{5/4}}.$$

(d)
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^3+50}}$$

Solution: Diverges by the limit comparison test and the divergence of the harmonic series, since
$$\lim_{n\to\infty} \frac{n}{\sqrt[3]{n^3+50}} = 1.$$

(e)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(\log n)^2}$$

Solution: Converges conditionally. It converges by the alternating series test, and the series of absolute values diverges by comparison to the harmonic series, since $(\log n)^2/n$ is bounded.

(15) 5. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by
$$y = e^x, y = e^{-x}, x = 1$$

about the y-axis.

Solution: The volume is

$$\int_0^1 2\pi x(e^x - e^{-x})dx = 4\pi e^{-1},$$

where the integration is done by integration by parts.

(10) 6. Find the value of c such that the area of the region enclosed by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

Solution: We need to choose $c > 0$ so that

$$2\int_{-c}^{c} (c^2 - x^2)dx = 576.$$

The left side above is $(8/3)c^3$, so $c = 6$.

(15) 7. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

Solution: Think of the centers of the two spheres as being at the points (0,0,0) and $(r, 0, 0)$. For $r/2 < x < r$, the cross-section corresponding to the value x is a disk with radius $\sqrt{r^2 - x^2}$. Therefore, the volume is given by

$$2\pi \int_{r/2}^{r} (r^2 - x^2) dx = 5r^3\pi/12.$$

(10) 8. Evaluate the following integrals:

(a) $\displaystyle\int \frac{\cos x}{2 + \sin x} dx$

Solution: Letting $u = \sin x$,

$$\int \frac{\cos x}{2 + \sin x} dx = \int \frac{du}{2 + u} = \ln|2 + u| + C = \ln(2 + \sin x) + C.$$

(b) $\displaystyle\int_0^2 x^3 \sqrt{x^2 + 4} dx$

Solution: There are at least two ways to do this one, each involving two substitutions. For the first, let $x = \tan\theta$, and then $u = \sec\theta$:

$$\int_0^2 x^3 \sqrt{x^2 + 4} dx = 32 \int_0^{\pi/4} \tan^3\theta \sec^3\theta d\theta = 32 \int_1^{\sqrt{2}} u^2(u^2 - 1) du = \frac{64}{15}(\sqrt{2} + 1).$$

For the second, let $u = x^2$, and then $v = u + 4$:

$$\int_0^2 x^3 \sqrt{x^2 + 4} dx = \frac{1}{2} \int_0^4 u\sqrt{u + 4} du = \frac{1}{2} \int_4^8 (v^{3/2} - 4v^{1/2}) dv = \frac{64}{15}(\sqrt{2} + 1).$$

(15) 9. Find the area of the surface obtained by rotating the curve

$$9x = y^2 + 18, \quad 2 \le x \le 6,$$

about the x-axis.
Solution: The area is

$$2\pi \int_2^6 \sqrt{9x - 18} \sqrt{1 + \frac{81}{4(9x - 18)}} dx = 3\pi \int_2^6 \sqrt{4x + 1} dx = 49\pi.$$

(10) 10. (a) Use properties of the logarithm to expand $\ln\sqrt{a(b^2 + c^2)}$.
Solution:

$$\ln\sqrt{a(b^2 + c^2)} = \frac{1}{2}\ln a(b^2 + c^2) = \frac{1}{2}\big(\ln a + \ln(b^2 + c^2)\big).$$

(b) Solve the equation $2\ln x = \ln 2 + \ln(3x-4)$ for x.
Solution: This can be written as $\ln x^2 = \ln(6x-8)$. Therefore $x^2 = 6x - 8$, or equivalently, $x = 2$ or $x = 4$.
(15) 11. (a) Find the radius of convergence and the interval of convergence for each of the two series

$$\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n}}$$

Solution: Use the ratio test:

$$L \lim_{n\to\infty} \frac{2^{n+1}|x|^{n+1}\sqrt{n}}{2^n|x|^n\sqrt{n+1}} = 2|x|,$$

so $R = 1/2$. When $x = 1/2$, the series converges by the alternating series test. When $x = -1/2$, the series diverges by the p-test. Therefore, the interval of convergence is $(-1/2, 1/2]$.

$$\sum_{n=1}^{\infty} \frac{n(x-4)^n}{n^3+1}$$

Solution: Similarly, $R = 1$, and the interval of convergence is $[3, 5]$.
(b) A function f is defined by

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + x^6 + \cdots.$$

Find the interval of convergence of the series, and find an explicit formula for $f(x)$.
Solution: Rewrite this as

$$f(x) = (1 + x^2 + x^4 + \cdots) + 2x(1 + x^2 + x^4 + \cdots) = (1+2x)\sum_{n=0}^{\infty}(x^2)^n.$$

Therefore, the interval of convergence is $(-1, 1)$, and

$$f(x) = \frac{1+2x}{1-x^2}.$$

(10) 12. By differentiating the geometric series $\sum_{n=0}^{\infty} x^n$, find the following sums explicitly:

(a) $$\sum_{n=0}^{\infty} nx^n$$

Solution:

(*)
$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} nx^{n-1},$$

so

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

(b) $\displaystyle\sum_{n=0}^{\infty} n^2x^n$

Solution: Differentiate (*) to get

$$\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} n(n-1)x^{n-2} = \frac{1}{x^2}\sum_{n=0}^{\infty} n^2x^n - \frac{1}{x^2}\sum_{n=0}^{\infty} nx^n.$$

Using the result of (a), we get

$$\sum_{n=0}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}.$$

(15) 13. (a) Find the Taylor series for $f(x) = 1/\sqrt{x}$ centered at $a = 9$.
Solution:

$$f(x) = x^{-1/2}, \quad f'(x) = -\frac{1}{2}x^{-3/2}, \quad f''(x) = \frac{1}{2}\frac{3}{2}x^{-5/2}, \quad f'''(x) = \frac{1}{2}\frac{3}{2}\frac{5}{2}x^{-7/2},$$

and in general,

$$f^{(n)}(x) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^{-(2n+1)/2}.$$

Therefore, the Taylor series is

$$f(x) = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!2^n3^{2n+1}}(x-9)^n = \frac{1}{3}\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{1}{36}\right)^n(x-9)^n.$$

(b) Express the following indefinite integral as an infinite series:

$$\int \frac{e^x-1}{x}dx$$

Solution:

$$\int \frac{e^x-1}{x}dx = \int \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}dx = \sum_{n=1}^{\infty}\frac{x^n}{nn!} + C.$$

(15) 14. Suppose f and g are one-to-one and twice differentiable functions that are inverses of each other.
(a) Express $g''(x)$ in terms of $f'(g(x))$ and $f''(g(x))$.
Solution: Start with $f(g(x)) = x$. Differentiate both sides to get $f'(g(x))g'(x) = 1$. Differentiate again to get

$$f''(g(x))[g'(x)]^2 + f'(g(x))g''(x) = 0.$$

Solving for $g''(x)$ and substituting for $g'(x)$ gives

$$g''(x) = -\frac{f''(g(x))[g'(x)]^2}{f'(g(x))} = -\frac{f''(g(x))}{[f'(g(x))]^3},$$

provided the denominator is not zero.
(b) Suppose f is decreasing and concave upward. What can you conclude about the concavity of g?
Solution: If f is decreasing and concave upward, then $f' \leq 0$ and $f'' \geq 0$. Therefore, $g'' \geq 0$, so g is concave upward.
(15) 15. (a) State the root test for convergence of series.
Solution: Suppose

$$L = \lim_{n\to\infty} |a_n|^{1/n}$$

exists. Then:

(a) if $L < 1$, $\sum a_n$ converges absolutely.
(b) if $L > 1$ (or the limit is infinite), then $\sum a_n$ diverges.
(c) if $L = 1$, either convergence or divergence can occur.

(b) Prove the root test for convergence of series.
Solution: (a) Suppose $L < 1$. Choose M so that $L < M < 1$. Then there is an N so that $n > N$ implies $|a_n|^{1/n} < M$, and therefore $|a_n| < M^n$. It follows that $\sum_n a_n$ converges absolutely by comparison with the geometric series $\sum M^n$.

(b) If $L > 1$, then there is an N so that $n > N$ implies that $|a_n|^{1/n} > 1$, and therefore $|a_n| > 1$. It follows that a_n does not converge to zero, and hence that $\sum a_n$ diverges.

(c) $L = 1$ if $a_n = 1/n^p$ for any p. Since this series converges if $p > 1$ and diverges if $p \leq 1$, we have examples of both convergence and divergence when $L = 1$.