Mathematics 275C – Stochastic Processes – T. Liggett

II. Markov Processes, Semigroups and Generators

References:

R. Blumenthal and R. Getoor, *Markov Processes and Potential Theory*, Academic Press, 1968.

S. Ethier and T. Kurtz, *Markov Processes: Characterization and Convergence*, Wiley, 1986.

T. Liggett, *Interacting Particle Systems*, Springer, 1985.

The Setting. The state space S of the process is a compact or locally compact metric space. Let $C(S)$ be the Banach space of continuous functions on S in the compact case, and of continuous functions that tend to zero at infinity in the locally compact space – with the sup norm: $||f|| = \sup_{x\in S} |f(x)|$. The main foundational results are that there are natural 1-1 correspondences among the following three objects:

A. Markov processes on S with the Feller property. Put $D[0,\infty)$ = the set of paths $\omega(\cdot)$ with values in S that are right continuous with left limits. The process is given by $X_t(\omega) = \omega(t)$. The natural filtration $\{\mathcal{F}_t, t \geq 0\}$ is given by $\mathcal{F}_t$ = the right continuous modification of the smallest σ-algebra on $D[0,\infty)$ with respect to which η_s is measurable for all $s \leq t$. For each $x \in S$, P^x is a probability measure on $D[0,\infty)$ that satisfies:

(a) $P^x\{X_0 = x\} = 1$.

(b) The mapping $x \to P^x(A)$ is measurable in x for each $A \in \mathcal{F}$.

(c) The Markov property: For bounded measurable Y, $E^x[Y \circ \theta_s \mid \mathcal{F}_s] = E^{X_s}Y$ a.s. (P^x) for every $x \in S, A \in \mathcal{F}$.

(d) The Feller property: For every $t \geq 0$, the mapping $f \to g$ where $g(x) = E^x f(X_t)$ maps $C(S)$ to $C(S)$.

Note: This implies the strong Markov property. The proof is the same as the one we saw for Brownian motion.

B. Strongly continuous semigroups of positive contractions. For each $t \geq 0$, $T(t)$ is an everywhere defined linear operator on $C(S)$. These satisfy:

(a) $T(0) = I$.

(b) Strong continuity: $T(t)f \to f$ as $t \downarrow 0$ for every $f \in C(S)$.

(c) The semigroup property: $T(t)T(s) = T(t+s)$ for $s, t \geq 0$.

(d) $T(t)f \geq 0$ whenever $f \geq 0$.

(e) $T(t)1 = 1$ for all $t \geq 0$ in the compact S case. In the locally compact case, the corresponding property is that $f_n \geq 0, f_n \uparrow 1$ pointwise implies $T(t)f_n \uparrow 1$ pointwise.

Note that (d) and (e) imply (f):

(f) The contraction property: $||T(t)f|| \leq ||f||$ for all f, t.

Connection between A and B:

$$T(t)f(x) = E^x f(X_t).$$

For the construction of the process from the semigroup, see pages 164-170 of Ethier and Kurtz (proof based on weak convergence and compactness), or pages 46-50 of Blumenthal and Getoor (proof based on martingale theory).

C. Semigroup generators. $\mathcal{L}$ is a (usually unbounded) linear operator on $C(S)$ with domain $\mathcal{D}(\mathcal{L}) \subset C(S)$. It satisfies:

(a) $\mathcal{D}(\mathcal{L})$ is dense in $C(S)$.

(b) $1 \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}1 = 0$ in the compact S case – an appropriate modification in the locally compact case.

(c) If $f \in \mathcal{D}(\mathcal{L}), \lambda \geq 0$ and $f - \lambda \mathcal{L} f = g$, then

$$\inf_x f(x) \geq \inf_x g(x).$$

Note: applying (c) to both f and $-f$ gives the following consequence: $||f|| \leq ||g||$. In other words,

(d) $(I - \lambda \mathcal{L})^{-1}$ is a contraction wherever it is defined.

(e) For all sufficiently small $\lambda > 0$, if $g \in C(S)$ there is an $f \in \mathcal{D}(\mathcal{L})$ so that $f - \lambda \mathcal{L} f = g$. In other words, $\mathcal{R}(I - \lambda \mathcal{L}) = C(S)$ for all sufficiently small positive λ.

Connection between B and C: In reading this, it is important to understand that this is a formalization of the relation

$$T(t) = e^{t\mathcal{L}},$$

which is correct only if $\mathcal{L}$ is a bounded operator. If one tries to define $T(t)$ by either

$$T(t)f = \sum_{n=0}^{\infty} \frac{(t\mathcal{L})^n f}{n!} \qquad \text{or} \qquad T(t)f = \lim_{n\to\infty} \left(I + \frac{t}{n}\mathcal{L}\right)^n f$$

in the unbounded case, one would immediately run into problems since these formulas only make sense for $f \in \cap_n \mathcal{D}(\mathcal{L}^n)$, and this can be a very small set of functions. First we will see how to get the generator from the semigroup, then how to get the semigroup from the generator, and finally will see other useful ways in which the two are related.

(a) $$\mathcal{D}(\mathcal{L}) = \left\{ f \in C(S) : \lim_{t\downarrow 0} \frac{T(t)f - f}{t} \text{ exists} \right\}; \quad \mathcal{L}f = \lim_{t\downarrow 0} \frac{T(t)f - f}{t}.$$

(b) $$T(t)f = \lim_{n\to\infty} \left(I - \frac{t}{n}\mathcal{L}\right)^{-n} f \qquad \text{for all } f \in C(S).$$

If $f \in \mathcal{D}(\mathcal{L})$, then $T(t)f \in \mathcal{D}(\mathcal{L})$ and

(c) $$\frac{d}{dt} T(t)f = \mathcal{L}T(t)f = T(t)\mathcal{L}f.$$

If $g \in C(S)$ and $\lambda \geq 0$, then the solution to $f - \lambda \mathcal{L} f = g$ is given by

(d) $$f = \int_0^\infty e^{-t} T(\lambda t) g dt.$$

The equivalence of parts (a), (b), (c), (f) of B to parts (a), (d), (e) of C is the Hille-Yosida theorem, and holds with $C(S)$ replaced by any Banach space. For a proof, see pages 10-20 of Ethier and Kurtz.

Markov pregenerators. Since $\mathcal{D}(\mathcal{L})$ is almost never known explicitly, the usual construction of $\mathcal{L}$ begins by constructing what is known as a "pregenerator", and then taking closures. A pregenerator satisfies all the properties of a generator except possibly C(e). Here is the basic fact that one uses to make the transition from a pregenerator to a generator: If a pregenerator $\mathcal{L}$ satisfies

(e') $\mathcal{R}(I-\lambda\mathcal{L})$ is dense in $C(S)$ for all sufficiently small positive λ,

then the (graph) closure of $\mathcal{L}$ is a generator. In this case, the domain of the original pregenerator is said to be a "core" for the resulting generator.

Comments. (a) In general, the graph closure of a linear operator may not be single valued. Consider for example the operator $\mathcal{L}$ defined by $\mathcal{L}f(x)=f'(0)$ with $S=[0,1]$ and $\mathcal{D}(\mathcal{L})=\{f\in C[0,1]: f'(0) \text{ exists}\}$. However, this difficulty does not occur with pregenerators. To see this, suppose $\mathcal{L}$ is a pregenerator, $f_n\in\mathcal{D}(\mathcal{L})$, $f_n\to 0$, and $\mathcal{L}f_n\to h$. Choose $g\in\mathcal{D}(\mathcal{L})$. By condition C(c),

$$||(I-\lambda\mathcal{L})(f_n+\lambda g||\geq ||f_n+\lambda g||,\quad \lambda>0.$$

Letting $n\to 0$, this implies that

$$||\lambda g-\lambda h-\lambda^2\mathcal{L}g||\geq||\lambda g||.$$

Dviding by λ and then letting $\lambda\downarrow 0$ gives $||g-h||\geq||g||$. Since $g\in\mathcal{D}(\mathcal{L})$ is arbitrary and $\mathcal{D}(\mathcal{L})$ is dense, it follows that $h=0$. This shows that the graph closure of $\mathcal{L}$ is single valued.

(b) A closed pregenerator $\mathcal{L}$ has closed $\mathcal{R}(I-\lambda\mathcal{L})$. To see this, take $g_n\in\mathcal{R}(I-\lambda\mathcal{L})$ so that $g_n\to g$. Define $f_n\in\mathcal{D}(\mathcal{L})$ by $f_n-\lambda\mathcal{L}f_n=g_n$. Then

$$(f_n-f_m)-\lambda\mathcal{L}(f_n-f_m)=g_n-g_m,$$

so that $||f_n-f_m||\leq||g_n-g_m||$ by property C(c). Since g_n is Cauchy, so is f_n, so f can be defined by $f=\lim_n f_n$. Therefore,

$$\lim_n\mathcal{L}f_n=\lambda^{-1}\lim_n(f_n-g_n)=\lambda^{-1}(f-g).$$

Since $\mathcal{L}$ is closed $f-\lambda\mathcal{L}f=g$, and $g\in\mathcal{R}(I-\lambda\mathcal{L})$ as required.

Example – Uniform motion to the right. Here $S=R^1$, $X_t=X_0+t$, $T(t)f(x)=f(x+t)$, and $\mathcal{L}f(x)=f'(x)$, with domain

$$\mathcal{D}(\mathcal{L})=\{f\in C(S): f'\in C(S)\}.$$

Example – Brownian motion. For homework, you will show that the generator of standard Brownian motion on $S=R^1$ is $\mathcal{L}f=\frac{1}{2}f''$ on

$$\mathcal{D}(\mathcal{L})=\{f\in C(S): f',f''\in C(S)\}.$$

One could instead start with this operator, and check that it satisfies the conditions in C above. For example, to check condition C(c), suppose $f - \lambda \frac{1}{2} f'' = g$, where $f, g \in C(S)$. If $f \geq 0$ on S, $\inf_x f(x) = 0 \geq \inf_x g(x)$. Otherwise, f achieves its absolute minimum, say at x_0. Then $f''(x_0) \geq 0$, so $\inf_x f(x) = f(x_0) \geq g(x_0) \geq \inf_x g(x)$ as required. To check C(d), suppose $g \in C(S)$. Then $f - \lambda \frac{1}{2} f'' = g$ can be explicitly solved for f, by using (d) of the "connections between B and C".

Example – The Cauchy process. This is the process on $S = R^1$ with stationary, independent increments $X_{s+t} - X_s$ with Cauchy density

$$\frac{1}{\pi} \frac{t}{t^2 + x^2}.$$

Take f to be C^2 with compact support, and ϕ odd, C^2 with compact support, satisfying $\phi'(0) = 1$. To compute $\mathcal{L} f$ for such an f, write

$$\frac{T(t)f(x) - f(x)}{t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x+y) - f(x) - f'(x)\phi(y)}{t^2 + y^2} dy$$

The term involving ϕ is added to make it possible to take the limit as $t \to 0$. Note that $f(x+y) = f(x) + y f'(x) + O(y^2)$ uniformly in x, so

$$f(x+y) - f(x) - f'(x)\phi(y) = f'(x)[y - \phi(y)] + O(y^2) = O(y^2)$$

uniformly in x. It follows that

$$\mathcal{L} f(x) = \lim_{t \downarrow 0} \frac{T(t)f(x) - f(x)}{t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x+y) - f(x) - f'(x)\phi(y)}{y^2} dy.$$

Note that, unlike the case of Brownian motion, this generator is not local. ($\mathcal{L}$ is local if $\mathcal{L} f(x)$ depends on the values of f only in a neighborhood of x.) As we know, the Cauchy process does not have continuous paths, while Brownian motion does. In general, continuity of paths corresponds to locality of $\mathcal{L}$.

Next we will note that there are many martingales associated with Markov processes.

Proposition 1. *Suppose X_t has semigroup $T(t)$ and generator $\mathcal{L}$. For every $f \in \mathcal{D}(\mathcal{L})$,*

$$M_t = f(X_t) - \int_0^t \mathcal{L} f(X_s) ds$$

is a martingale relative to P^x for every x.

Proof. First compute

$$E^x M_t = T(t)f(x) - \int_0^t T(s) \mathcal{L} f(x) ds = T(t)f(x) - \int_0^t \frac{d}{ds} T(s) f(x) ds = f(x).$$

Therefore, for $s < t$, the Markov property gives

$$\begin{aligned} E(M_t \mid \mathcal{F}_s) =& E\left(f(X_{t-s}) \circ \theta_s - \int_0^s \mathcal{L} f(X_u) du - \int_0^{t-s} \mathcal{L} f(X_u) \circ \theta_s du \middle| \mathcal{F}_s \right) \\ =& E^{X_s} f(X_{t-s}) - \int_0^s \mathcal{L} f(X_u) du - E^{X_s} \int_0^{t-s} \mathcal{L} f(X_u) du \\ =& f(X_s) - \int_0^s \mathcal{L} f(X_u) du = M_s. \end{aligned}$$

Here is an example to show how these martingales can be used.

Example – The Fisher-Wright diffusion. This is a model for the evolution of gene frequencies, so we take $S = [0,1]$. When restricted to an appropriate subset of $C^2[0,1]$, the generator is given by

$$\mathcal{L}f(x) = \frac{1}{2}x(1-x)f''(x).$$

The interpretation is that if a population consists of A's and a's with frequencies x and $1-x$ respectively and new individuals appear when two parents chosen randomly from the population mate, $2x(1-x)$ is the probability that the two parents are of opposite type, and in this case the offspring is of either type with probability $\frac{1}{2}$. So, the evolution of the gene frequency should be Brownian, but with a rate that is proportional to $x(1-x)$. To analyze the process, consider the martingales produced by Proposition 1 for the following choices of f:

(a) $f(x) = x$. Then $\mathcal{L}f = 0$, so X_t itself is a martingale. Since it is bounded,

$$X_\infty = \lim_{t\to\infty} X_t \quad \text{exists, and} \quad E^x X_\infty = x. \tag{1}$$

(b) $f(x) = x(1-x)$. Now $\mathcal{L}f(x) = -x(1-x)$, so

$$X_t(1-X_t) + \int_0^t X_s(1-X_s)ds$$

is a martingale. Therefore, since the limit of this martingale is finite, it follows that

$$P^x(X_\infty = 0 \text{ or } 1) = 1,$$

and then by (1), $P^x(X_\infty = 1) = x$. Furthermore, using the above martingale and letting $t \to \infty$ gives

$$E^x \int_0^\infty X_s(1-X_s)ds = x(1-x).$$

(c) $f(x) = 2x\log x + 2(1-x)\log(1-x)$. Formally, $\mathcal{L}f \equiv 1$. However, $f \notin C^2[0,1]$, and it follows from Proposition 1 that $f \notin \mathcal{D}(\mathcal{L})$, since if it were in the domain, the corresponding martingale would tend to $-\infty$ as $t \to \infty$. To fix this, take $f_\epsilon \in C^2[0,1]$ so that $f_\epsilon = f$ on $[\epsilon, 1-\epsilon]$. Let τ_ϵ be the hitting time of $\{\epsilon, 1-\epsilon\}$. Since

$$f_\epsilon(X_t) - \int_0^t \mathcal{L}f_\epsilon(X_s)ds$$

is a martingale, $f(X_{\tau_\epsilon \wedge t}) - \tau_\epsilon \wedge t$ is a P^x martingale if $\epsilon < x < 1-\epsilon$. It follows that

$$E^x f(X_{\tau_\epsilon}) - E^x \tau_\epsilon = f(x), \quad \epsilon < x < 1-\epsilon,$$

and then that $E^x\tau = -f(x)$ for $x \in (0,1)$, where τ is the hitting time of $\{0,1\}$. Note that if X_t were Brownian motion, then $E^x\tau = x(1-x)$.

As we saw above, the issue of exactly what is in $\mathcal{D}(\mathcal{L})$ is unclear. To construct the process via the Hille-Yosida Theorem in the above example, we define $\mathcal{L}f$ for reasonable f's, and recalling our discussion of pregenerators, need to show that

$\mathcal{R}(I-\lambda\mathcal{L})$ is dense in $C[0,1]$. To do so, take a polynomial $g(x)=\sum_{k=0}^n a_k x^k$, and try to solve the equation $f-\lambda\mathcal{L}f=g$ for f. Write

$$f(x)=\sum_{k=0}^n c_k x^k.$$

The equation $f-\lambda\mathcal{L}f=g$ then becomes

$$c_k-\frac{\lambda}{2}\big[k(k+1)c_{k+1}-(k-1)kc_k\big]=a_k,$$

which can be solved recursively for the c_k's. So, if we define $\mathcal{L}$ originally on the set of polynomials, we will have that $\mathcal{R}(I-\lambda\mathcal{L})$ is dense in $C[0,1]$ as required. It follows that the domain of the actual generator is

$$\{f: \text{ there exists polynomials } p_n \text{ such that } p_n\to f \text{ uniformly and } \\ x(1-x)p_n'' \text{ has a uniform limit}\}.$$

In particular, every $f\in\mathcal{D}(\mathcal{L})$ satisfies $\mathcal{L}f(0)=\mathcal{L}f(1)=0$, so we see again that the f in (c) above is not in $\mathcal{D}(\mathcal{L})$.

The following proposition shows how one can determine stationary distributions for the process by looking at the generator. A probability measure μ is said to be stationary if

$$\int T(t)f\,d\mu=\int f\,d\mu \qquad \text{for all } f\in C(S), t>0.$$

Note that the left side above is

$$\int E^x f(X_t)\mu(dx)=E^\mu f(X_t),$$

so stationarity just means that if the initial distributiion is μ, then the distribution of the process is μ at all later times.

Proposition 2. *Suppose D is a core for the generator $\mathcal{L}$. Then a probability measure μ on S is stationary for the corresponding process if and only if*

$$\int \mathcal{L}f\,d\mu=0 \quad \textit{for all } f\in D.$$

Proof. Suppose μ is stationary, and take $f\in\mathcal{D}(\mathcal{L})$. Then

$$\int\mathcal{L}f\,d\mu=\int\lim_{t\downarrow 0}\frac{T(t)f-f}{t}d\mu=\lim_{t\downarrow 0}\frac{\int T(t)f\,d\mu-\int f\,d\mu}{t}=0.$$

Conversely, suppose $\int\mathcal{L}f\,d\mu=0$ for all $f\in D$. If $f\in\mathcal{D}(\mathcal{L})$, then there are $f_n\in D$ so that $f_n\to f$ and $\mathcal{L}f_n\to\mathcal{L}f$. Therefore, $\int\mathcal{L}f\,d\mu=0$ for all $f\in\mathcal{D}(\mathcal{L})$. If $f\in\mathcal{D}(\mathcal{L})$ and $f-\lambda\mathcal{L}f=g$, then $\int f\,d\mu=\int g\,d\mu$. Therefore,

$$\int(I-\lambda\mathcal{L})^{-1}g\,d\mu=\int g\,d\mu, \qquad \text{for all } g\in C(S),\lambda\geq 0.$$

It follows that

$$\int T(t)g\,d\mu = \lim_{n\to\infty}\int \left(I - \frac{t}{n}\mathcal{L}\right)^{-n} g\,d\mu = \int g\,d\mu,$$

so that μ is stationary.

Example – Brownian motion on $S = R^1$ with speed change. Suppose $c(x) > 0$ is continuous, and is such that $\mathcal{L}f(x) = c(x)f''(x)$ is a generator with core = the set of C^2 functions with compact support. If $\int 1/c(x)dx < \infty$, then $[c(x)]^{-1}dx$ is stationary for the process. To check uniqueness of this stationary distribution, we proceed as follows. All functions (other than $c(x)$) below are assumed to be continuous and have compact support. First check that $g = f''$ can be solved for f iff $\int g(x)dx = \int xg(x)dx = 0$. Therefore, if μ is stationary, $\int c(x)g(x)d\mu = 0$ for all such g. For any function g, the function $g(x) - (ax+b)h(x)$ satisfies these conditions if $h(x)$ is any function satisfying

$$\int h(x)dx = \int x^2h(x)dx = 0, \qquad \int xh(x)dx = 1,$$

and

$$a = \int g(x)dx, \quad b = \int xg(x)dx.$$

Therefore, for any function $g(x)$,

$$\int c(x)g(x)d\mu = \int (ax+b)h(x)d\mu.$$

This can be rewritten as

$$\int c(x)g(x)d\mu = \int (A+Bx)g(x)dx,$$

where

$$A = \int xc(x)h(x)d\mu, \quad B = \int c(x)h(x)d\mu.$$

It follows that $c(x)d\mu = (A+Bx)dx$. Since μ is a measure (not a signed measure), $B = 0$, so that $c(x)d\mu = Adx$ as required.

Next we consider how boundary behavior enters into the description of the generator.

Example – Brownian motion on $S = [0,\infty)$ with absorbing or reflecting boundary at 0. Suppose B_t is Brownian motion on R^1. Define the process on $[0,\infty)$ with reflecting boundary at 0 by $X_t = |B_t|$ and the process with absorbing boundary at 0 by

$$Y_t = \begin{cases} B_t & \text{if } t < \tau, \\ 0 & \text{if } t \geq \tau, \end{cases}$$

where τ is the hitting time of 0. It shouldn't be surprising that $\mathcal{L}f = \frac{1}{2}f''$ in both cases, if f is in the corresponding domain. To check this, suppose f is in the domain, and for $x > 0$ write (say for X_t),

$$\frac{E^xf(X_t) - f(x)}{t} = \frac{E^xf(B_t) - E^x[f(B_t), \tau \leq t] + E^x[f(X_t), \tau \leq t] - f(x)}{t}.$$

By the reflection principle, letting $M_t = \max_{s\le t} B_s$,

$$P^x(\tau \le t) = P^0(M_t > x) = 2P^0(B_t > x) = 2P^0(B_1 > x/\sqrt{t}) = o(t), \quad t \downarrow 0.$$

Therefore,

$$\lim_{t\downarrow 0} \frac{E^x f(X_t) - f(x)}{t} = \lim_{t\downarrow 0} \frac{E^x f(B_t) - f(x)}{t} = \frac{1}{2} f''(x), \quad x > 0.$$

The processes X_t and Y_t are clearly not the same, so the two generators must have different domains. The question is, what are the two domains? To answer this question, note that for a Borel set $A \subset (0,\infty)$,

$$P^x(X_t \in A) = P^x(B_t \in A) + P^x(B_t \in -A) \tag{2}$$

and by the reflection principle,

$$P(Y_t \in A) = P^x(B_t \in A, \tau > t) = P^x(B_t \in A) - P^x(B_t \in -A). \tag{3}$$

Let the semigroups and generators of the processes B_t, X_t and Y_t be $T(t), T_r(t), T_a(t)$ and $\mathcal{L}, \mathcal{L}_r, \mathcal{L}_a$ respectively.

The generator $\mathcal{L}_r$. For f on $[0,\infty)$, define the even extension f_e of f to be the function on R^1 that satisfies $f_e(x) = f_e(-x)$ and $f_e(x) = f(x)$ for $x \ge 0$. Then $T_r(t)f(x) = T(t)f(x)$ by (2), so that $f \in \mathcal{D}(\mathcal{L}_r)$ if and only if $f_e \in \mathcal{D}(\mathcal{L})$. For smooth t, this will only be the case if $f'(0) = 0$. In fact,

$$\mathcal{D}(\mathcal{L}_r) = \{f \in C[S] : f', f'' \in C(S), f'(0) = 0\}.$$

The generator $\mathcal{L}_a$. If $f(0) = 0$, define the odd extension of f to R^1 by $f_o(x) = -f_o(-x)$ and $f_o(x) = f(x)$ for $x \ge 0$. By (3), $T_a(t)f(x) = T(t)f_o(x)$. Since $\mathcal{L}_a f(0) = 0$, and $\mathcal{L}_a f(x) = \frac{1}{2}f''(x)$ for $x > 0$, $f \in \mathcal{D}(\mathcal{L}_a)$ implies that $f''(0) = 0$. In fact,

$$\mathcal{D}(\mathcal{L}_r) = \{f \in C[S] : f', f'' \in C(S), f''(0) = 0\}.$$

Remark. Note that $\mathcal{D}(\mathcal{L}_r) \cap \mathcal{D}(\mathcal{L}_a)$ is dense in $C(S)$, and $\mathcal{L}_r = \mathcal{L}_a$ on $\mathcal{D}(\mathcal{L}_r) \cap \mathcal{D}(\mathcal{L}_a)$. Therefore, the restriction of a generator to a dense set does not determine the process uniquely.

Example – Brownian motion on $S = [0,\infty)$ with a "sticky" boundary at 0. Consider $\mathcal{L}f = \frac{1}{2}f''$ on

$$\mathcal{D}(\mathcal{L}) = \{f \in C[S] : f', f'' \in C(S), f'(0) = cf''(0)\}$$

for a fixed $c > 0$. This boundary condition interpolates between the reflecting and absorbing cases. The verification of condition C(c) is the same as it was for Brownian motion on R^1, except in case $x_0 = 0$. In this case, we know from the boundary condition that $f'(0)$ and $f''(0)$ have the same sign. Therefore $f''(0) \ge 0$ as before, since if $f''(0) < 0$, then $f'(0) \le 0$, which contradicts the assumption that

f has its minimum at 0. Using relation (d) from “relations between B abd C”, one can check that

$$E^0 \int_0^\infty \alpha e^{-\alpha t} 1_{\{X_t > 0\}} dt = \frac{1}{1 + c\sqrt{2\alpha}}, \tag{4}$$

so that the Lebesgue measure of $\{t : X_t = 0\}$ is positive, unlike the reflecting boundary case (which corresponds to $c = 0$).

Here is the idea to check (4). Since the semigroups are known explicitly in the absorbing and reflecting case, the following equations can be solved explicitly for $f_a \in \mathcal{D}(\mathcal{L}_a)$ and $f_r \in \mathcal{D}(\mathcal{L}_r)$:

$$f_a - \lambda \mathcal{L}_a f_a = g, \quad f_r - \lambda \mathcal{L}_r f_r = g.$$

Since the form of the generators are all the same – only the domains differ, one can then solve $f - \lambda \mathcal{L} f = g$ by taking f to be a constant multiple of $f_r'(0) f_a(x) + c f_a''(0) f_r(x)$. That provides an expression for

$$E^x \int_0^\infty \alpha e^{-\alpha t} g(X_t) dt = \int_0^\infty \alpha e^{-\alpha t} T(t) g(x) dt.$$