

### Comments on HW #4

1. Many students interchanged derivatives and sums without justification. In fact, it is not easy to justify this without changing the differential equations into integral equations.

Using the technique from class, the system of differential equations, together with the initial condition, is equivalent to

$$(*) \quad u(t, x) = \phi(x)e^{-tc(x)} + \int_0^t e^{-(t-s)c(x)} \sum_{z \neq x} q(x, z)u(s, z)ds.$$

To check that the given  $u(t, x)$  is a solution, multiply (6) from the notes by  $\phi(y)$  and sum, using DCT to justify the various interchanges.

For uniqueness, one must show that if  $u(t, x)$  is any bounded solution with  $\phi \equiv 0$ , then  $u \equiv 0$ . Suppose  $u(t, x)$  is such a solution and satisfies  $u(t, x) \geq M$ . Using (\*) above, one gets

$$u(t, x) \geq M \left( 1 - \sum_y p_t^{(0)}(x, y) \right).$$

Using (\*) again, one gets inductively

$$u(t, x) \geq M \left( 1 - \sum_y p_t^{(n)}(x, y) \right) \quad n \geq 0.$$

Passing to the limit on  $n$  gives

$$u(t, x) \geq M \left( 1 - \sum_y p_t(x, y) \right),$$

where  $p_t(x, y)$  is the minimal solution to the BE. Since it is assumed to be stochastic, it follows that  $u(t, x) \geq 0$ . A similar argument gives  $u(t, x) \leq 0$ .

2. By scaling,

$$\frac{T(t)f(x) - f(x)}{t} = \frac{1}{t} \int [f(x + \sqrt{t}y) - f(x)]p_1(0, y)dy.$$

By Taylor's theorem with remainder,

$$\frac{T(t)f(x) - f(x)}{t} - \frac{1}{2}f''(x) = \frac{1}{2} \int y^2 [f''(u) - f''(x)]p_1(0, y)dy,$$

where  $u = u(x, t, y)$  lies between  $x$  and  $x + \sqrt{t}y$ . Since  $f''$  is continuous and tends to 0 at  $\pm\infty$ , it is uniformly continuous on  $R$ . Let  $\delta(\epsilon)$  be its modulus of continuity. Then

$$\sup_x \left| \frac{T(t)f(x) - f(x)}{t} - \frac{1}{2}f''(x) \right| \leq \frac{1}{2} \int y^2 \delta(\sqrt{t}y)p_1(0, y)dy,$$

which tends to 0 by the DCT.

3. The NASC in (b) is  $\sum 1/\delta(n) = \infty$ , which most students got. Most thought that this was also the NASC for the minimal solution to be stochastic, but this is not correct. The minimal solution is always stochastic, by the third criterion in Theorem 3 from the notes. One of the points of this problem is that the BE approach to continuous time Markov chains applies more generally than the semigroup approach based on the Hille-Yosida theorem.