Mathematics 275C – Spring 2008 – HW #4

Due: Wednesday, May 28.

1. Suppose that the Q-matrix for a Markov chain on the countable set S satisfies $q(x, y) \geq 0$ for all $y \neq x$ and $\sum_y q(x, y) = 0$ for all x. Put $c(x) = -q(x, x)$. Let $p_t(x, y)$ be the minimal solution to the backward equation, and assume that it is stochastic and satisfies the Chapman-Kolmogorov equations (which is automatically true). Given a bounded function ϕ on S, prove that the unique bounded solution to

$$u'(t, x) = \sum_z q(x, z)u(t, z), \quad u(0, x) = \phi(x)$$

is given by

$$u(t, x) = \sum_y p_t(x, y)\phi(y).$$

2. Suppose f, f', f'' are continuous on R^1 and tend to zero at $\pm\infty$, and let $T(t)$ be the semigroup for standard one-dimensional Brownian motion:

$$T(t)f(x) = \int_{-\infty}^{\infty} p_t(x, y)f(y)dy,$$

where

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}.$$

Show directly that

$$\lim_{t \downarrow 0} \sup_x \left| \frac{T(t)f(x) - f(x)}{t} - \frac{1}{2} f''(x) \right| = 0,$$

so that $f \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}f = \frac{1}{2} f''$. (Suggestion: Use scaling to write everything in terms of integrals with respect to the function $p_1(0, \cdot)$.)

3. Consider the birth and death chain on $S = \{0, 1, ...\}$ with Q-matrix

$$q(x, x + 1) = \beta(x), \quad q(x, x - 1) = \delta(x), \quad q(x, x) = -[\beta(x) + \delta(x)].$$

($\delta(0) = 0$.) Define the formal generator by

$$\mathcal{L}f(x) = \sum_y q(x, y)f(y), \quad \mathcal{D}(\mathcal{L}) = \{f \in c_0(S) : \mathcal{L}f \in c_0(S)\},$$

where $c_0(S)$ is the set of functions on S that tend to 0 at infinity. Note that $\mathcal{D}(\mathcal{L})$ is dense in $c_0(S)$.

(a) Show that $f \in \mathcal{D}(\mathcal{L}), \lambda > 0, f - \lambda\mathcal{L}f = g \geq 0$ implies $f \geq 0$.

(b) Now assume $\beta(0) = 1$ and $\beta(x) = 0$ and $\delta(x) > 0$ for all $x \geq 1$. Find a necessary and sufficient condition on the $\delta(x)$'s so that for every g that is nonzero for only finitely many x, there is a solution $f \in c_0(S)$ to $f - \lambda\mathcal{L}f = g$ for $\lambda > 0$. How does this condition compare to the condition for the minimal solution to the backward equation $p_t(x, y)$ to be stochastic? When your condition fails, what can you say about

$$\lim_{x \to \infty} p_t(x, y)?$$