Mathematics 275C – Spring 2008 – HW #1 – (Partial) Solutions

1. Here is a proof for general k. For $k = 1, 2$, things simplify significantly, of course.

$$\begin{aligned} E\left[\frac{1}{\sqrt{t}}\int_0^t f(B(s))ds\right]^k =& \frac{1}{t^{k/2}} E\int_0^t \cdots \int_0^t f(B(s_1))\cdots f(B(s_k))ds_1\cdots ds_k \\ =& k!t^{k/2}\int\cdots\int 1_{\{0<r_1<\cdots<r_k<1\}} Ef(B(tr_1))\cdots f(B(tr_k))dr_1\cdots dr_k \\ =& \frac{k!}{(2\pi)^{k/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(y_1)f(y_1+y_2)\cdots f(y_1+\cdots+y_k)\phi dy_1\cdots dy_k, \end{aligned}$$

where

$$\phi = \phi(t, y_1, ..., y_k) = \int\cdots\int 1_{\{0<r_1<\cdots<r_k<1\}} \frac{e^{-\frac{y_1^2}{2tr_1}}\cdots e^{-\frac{y_k^2}{2t(r_k-r_{k-1})}}}{\sqrt{r_1}\cdots\sqrt{r_k-r_{k-1}}} dr_1\cdots dr_k.$$

To justify the interchange of integration above, use Tonelli with f replaced by $|f|$, and then (since the right side is finite) Fubini.

Since $\phi(t, y_1, ..., y_k) \le \phi(t, 0, ..., 0)$ and $\phi(t, y_1, ..., y_k) \to \phi(1, 0, ..., 0)$ as $t \to \infty$, it follows from the dominated convergence theorem that

$$\lim_{t\to\infty} E\left[\frac{1}{\sqrt{t}}\int_0^t f(B(s))ds\right]^k = \frac{k!}{(2\pi)^{k/2}}\int\cdots\int \frac{1_{\{0<r_1<\cdots<r_k<1\}}}{\sqrt{r_1}\cdots\sqrt{r_k-r_{k-1}}} dr_1\cdots dr_k. \tag{1}$$

Integration by parts shows that the moments of $|Z|$ satisfy the recursion

$$E|Z|^k = (k-1)E|Z|^{k-2}, \quad k \ge 2,$$

with initial condition $E|Z| = \sqrt{2/\pi}, EZ^2 = 1$. So, it suffices to show that the limits in (1) satisfy the same recursion and initial condition. To do so, define

$$I(\alpha_1, ..., \alpha_k) = \int\cdots\int 1_{\{0<r_1<\cdots<r_k<1\}} r_1^{\alpha_1-1}(r_2-r_1)^{\alpha_2-1}\cdots(r_k-r_{k-1})^{\alpha_k-1}dr_1\cdots dr_k$$

for $\alpha_1, ..., \alpha_k > 0$. Using the Beta integral

$$\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}du = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)},$$

one gets the recursion

$$I(\alpha_1, ..., \alpha_k) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} I(\alpha_1+\alpha_2, \alpha_3, ..., \alpha_k),$$

and hence

$$I(\alpha_1, ..., \alpha_k) = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)}{\Gamma(\alpha_1 + \cdots + \alpha_k)} I(\alpha_1 + \alpha_2 + \cdots + \alpha_k).$$

Since $I(\alpha) = 1/\alpha$, one gets

$$I\left(\frac{1}{2}, ..., \frac{1}{2}\right) = \frac{[\Gamma(1/2)]^k}{\Gamma(k/2)(k/2)},$$

where k is the number of arguments in I. Now use $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ and $\Gamma(1/2) = \sqrt{\pi}$ to complete the argument.

2(a) Since the left limits exist a.s., we may put $Z = \lim_{s \uparrow t} X(s)$. As $s \uparrow t$, $X(s) \to X(t)$ in probability, since

$$P(|X(s) - X(t)| > \epsilon) = P(|X(t - s)| > \epsilon),$$

and the right side tends to zero for each ϵ. This last statement follows from either the right continuity of paths, or from the fact that the characteristic function of $X(t - s)$ tends to 1, so $X(t-s)$ tends to zero in distribution, and hence in probability (since the limit is constant). Since $X(s)$ converges to both $X(t)$ and Z in probability, it follows that $X(t) = Z$ a.s.

2(c). Let $M_n = \max_{1 \leq k \leq n} |X(k/n) - X((k - 1)/n)|$. From part (b), we know that

$$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} P(M_n \geq \epsilon) = 1.$$

Let $A = \{\omega : t \to X(t, \omega)$ is continuous on $[0, 1]\}$. By uniform continuity, $M_n \to 0$ a.s. on A. Therefore $M_n 1_A \to 0$ a.s., and hence in probability. Writing

$$P(M_n \geq \epsilon) \leq P(M_n \geq \epsilon, A) + P(A^c),$$

and letting $n \to \infty$ first, and then $\epsilon \downarrow 0$, we see that $P(A^c) = 1$ as required.