Mathematics 275C – Spring 2008 – HW #1

Due: Wednesday, April 16.

1. Let $B(t)$ be standard Brownian motion, and take $f \in L_1(R)$ with $\int f(x)dx = 1$. The method of moments can be used to show that

$$\frac{1}{\sqrt{t}} \int_0^t f\big(B(s)\big)ds \to |Z|$$

in distribution, where Z is $N(0,1)$. Note that if $f = 1_A$ for some set A of Lebesgue measure 1, this gives the limiting distribution of the occupation time of A up to time t.

Since the moments of $|Z|$ uniquely determine its distribution (by Theorem 3.11 on page 108 of Durrett), this requires that one show that

$$(1) \qquad \lim_{t\to\infty} E\left[\frac{1}{\sqrt{t}} \int_0^t f\big(B(s)\big)ds\right]^k = E|Z|^k, \quad k = 1, 2, 3, ...$$

Prove (1) for $k = 1, 2$.

2. Suppose $\{X(t), t \geq 0\}$ is a stochastic process with stationary and independent increments, and that $X(t)$ has a symmetric stable distribution with index $\alpha \in (0, 2]$. Specifically, the characteristic function of $X(t)$ is

$$Ee^{iuX(t)} = e^{-t|u|^\alpha}.$$

Note that if $\alpha = 2$, then $X(t)$ is Brownian motion. For general $\alpha \in (0, 2]$, $X(t)$ is called a symmetric stable process. If $\alpha < 2$ the distribution of $X(1)$ satisfies $P(|X(1)| \geq x) \geq Cx^{-\alpha}$ for $x \geq 1$. (See Exercise 7.5 on page 159 of Durrett.) Note that this is false for $\alpha = 2$. Assume that with probability 1, the paths $X(t, \omega)$ are right continuous and have left limits. (Such paths are called cadlag, which is an acronym from the French phrase describing these properties; it is known that there is a cadlag version of this process.)

(a) Show that for each fixed t, $X(s)$ is a.s. continuous at $s = t$, i.e., X has no fixed times of discontinuity.

(b) Express

$$P\left(\max_{1\leq k\leq n}\left|X\left(\frac{k}{n}\right) - X\left(\frac{k-1}{n}\right)\right| \geq \epsilon\right)$$

in terms of the distribution of $X(1)$.

(c) Using (b), prove that for $\alpha \in (0, 2)$,

$$P(X(t) \text{ is continuous on } [0, 1]) = 0.$$

Note the contrast with Brownian motion. Rather few continuous time Markov processes have continuous paths; most have cadlag paths.

3. In the proof that Brownian paths are nowhere differentiable a.s. (to be done in class about April 9), we use three increments. Use more increments to show that w.p.1, if $\alpha > \frac{1}{2}$, Brownian paths satisfy a Hölder condition with exponent α at no time t. This means that

$$P\left(\exists t > 0 : \limsup_{s \to t} \frac{|B(s) - B(t)|}{|s - t|^\alpha} < \infty\right) = 0.$$

Note that this is stronger than the assertion that for each $t > 0$, the paths do not satisfy a Hölder condition with exponent α. Recall that in connection with the construction of Brownian motion as a process with continuous paths, we proved that if $\alpha < \frac{1}{2}$, then

$$P\left(\forall t > 0 : \limsup_{s \to t} \frac{|B(s) - B(t)|}{|s - t|^\alpha} < \infty\right) = 1.$$