

III. Brownian Motion and the Dirichlet Problem

References:

- J. L. Doob, *Classical Potential Theory and its Probabilistic Counterpart*, Springer, 1984.
 K. Ito and H. P. McKean, *Diffusion Processes and their Sample Paths*, Springer, 1965.
 S. C. Port and C. J. Stone, *Brownian motion and Classical Potential Theory*, Academic Press, 1978.

A little history. The main names associated with the early development of this subject are Kakutani (1944), Kac (1951), Doob (1954) and Hunt (1958). Not surprisingly, the war delayed communications between Japan and the US.

The Dirichlet Problem. Given a domain $D \subset R^d$ (open, connected) and a function f on ∂D , find a harmonic function h on D so that $h = f$ on ∂D , in the sense that

$$\lim_{\substack{x \rightarrow z \\ x \in D}} h(x) = f(z), \quad z \in \partial D.$$

Definition. A function h is harmonic on D if one of the following equivalent conditions holds:

(a) h is continuous on D , and for every $x \in D, r > 0$ such that the closed ball $B(x, r)$ with center x and radius r is contained in D ,

$$(1) \quad h(x) = \int_{\partial B(x,r)} h(y) \sigma(dy),$$

where σ is normalized surface measure with total mass 1. This is called the mean value property.

(b) $h \in C^2(D)$ and $\Delta h = 0$ on D , where Δ is the Laplacian:

$$\Delta h = \sum_{i=1}^d \frac{\partial^2 h}{\partial x_i^2}.$$

The proof of the equivalence of these conditions is analytic, using the divergence theorem.

Connection with the heat equation. Recall the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u,$$

which describes the evolution of temperature on D . If $h(x) = u(t, x)$ is a solution to the heat equation that does not depend on t , it is harmonic in D , and is a solution to the Dirichlet problem with fixed temperature f on ∂D .

Connection with Brownian motion $X(t)$. Brownian motion on R^d is just

$$X(t) = (B_1(t), \dots, B_d(t)),$$

where $B_1(t), \dots, B_d(t)$ are independent one dimensional Brownian motions. An important property of $X(t)$ is that it is rotationally invariant. This is a consequence of the fact that the density of $X(t)$ starting at 0, say, is just a function of $\|x\|$.

By an argument similar to the one done for homework, the generator of $X(t)$, when restricted to C^2 functions f so that f, f', f'' tend to zero sufficiently rapidly at ∞ , is $\frac{1}{2}\Delta f$. Therefore, for such functions, $u(t, x) = T(t)f(x)$ satisfies the heat equation.

For a more probabilistic connection with the Dirichlet problem itself, let

$$\tau_D = \inf\{t > 0 : X(t) \in D^c\},$$

and for $f \geq 0$ on ∂D , define

$$h(x) = E^x[f(X(\tau_D), \tau_D < \infty)].$$

Theorem 1. *If h is not identically infinity on D , then h is harmonic on D .*

Proof. Put $B = B(x, r) \subset D$. Then $\tau_B < \infty$ a.s., as can be seen by considering a single coordinate of X_t . Since for paths beginning inside B , $X(\tau_D) = X(\tau_D) \circ \theta_{\tau_B}$, and $\tau_D \circ \theta_{\tau_B} < \infty$ iff $\tau_D < \infty$, the strong Markov property gives

$$E^x[f(X(\tau_D)), \tau_D < \infty \mid \mathcal{F}_{\tau_B}] = E^{X(\tau_B)}[f(X(\tau_D), \tau_D < \infty)], \quad a.s. P^x$$

for $x \in B$. Taking expected values gives

$$(2) \quad h(x) = E^x h(X(\tau_B)), \quad x \in B.$$

By rotational invariance of $X(t)$, $X(\tau_B)$ is uniformly distributed on $\partial B(x, r)$. Therefore (2) is just the mean value property (1).

Of course, (2) may just say $\infty = \infty$. To rule this out when h is not identically infinity, we proceed as follows. Multiply (2) by a function of r and integrate, to see that

$$(3) \quad h(x) = \frac{1}{m(B)} \int_B h(y) dy$$

for any closed ball $B \subset D$. (m denotes Lebesgue measure on R^d .) It follows that if $x_1, x_2 \in B \subset D$, then $h(x_1) \leq C(\epsilon)h(x_2)$ if x_1, x_2 are a distance at least ϵ from ∂B . (This is known as Harnack's inequality.) So, $\{x : h(x) < \infty\}$ is both open and closed. Since D is connected, either $h \equiv \infty$ on D , or it is finite everywhere on D . To check that h is continuous on D , use (3) to write

$$|h(x_1) - h(x_2)| \leq \frac{1}{m(B(r))} \int_{B(x_1, r) \Delta B(x_2, r)} h(y) dy.$$

Here are some questions that we would like to address:

- (a) Is $h = f$ on ∂D ?
- (b) Is h the unique solution to the Dirichlet problem?
- (c) How can we use solutions to the Dirichlet problem to obtain information about Brownian motion on R^d ?

A very useful tool in studying the uniqueness question is the *maximum principle*:

Theorem 2. *If a harmonic function h achieves its maximum or minimum on D , then it is constant on D .*

Proof. Suppose $M = \max_{x \in D} h(x)$ is achieved in D . By (3), $h(x) = M$ implies $h \equiv M$ in a neighborhood of x . Therefore $\{x \in D : h(x) = M\}$ is both open and closed, and hence $h \equiv M$ on D .

Corollary. *Suppose D is bounded, h_1 and h_2 are both harmonic on D , continuous on the closure of D , and $h_1 = h_2$ on ∂D . Then $h_1 \equiv h_2$ on D .*

Proof. Put $h = h_1 - h_2$. Then h is harmonic on D , continuous on the closure of D , and $= 0$ on ∂D . If $h \not\equiv 0$ on D , then it achieves a strictly positive maximum or strictly negative minimum on D , and therefore is constant on D , which is a contradiction.

Examples. Using condition (b) in the definition of harmonicity, it is easy to check that

- (a) $h(x) = x$ is harmonic on R^1 ,
- (b) $h(x) = \log \|x\|$ is harmonic on $R^2 \setminus \{0\}$, and
- (c) $1/\|x\|^{d-2}$ is harmonic on $R^d \setminus \{0\}$ if $d \geq 3$.

Applications. Take $0 < r_1 < r_2$ and $D = \{x \in R^d : r_1 < \|x\| < r_2\}$. By Theorem 1, letting $f(x) = 0$ for $\|x\| = r_1$ and $f(x) = 1$ for $\|x\| = r_2$,

$$h_1(x) = P^x(X(t) \text{ hits } \partial B(0, r_2) \text{ before } \partial B(0, r_1)) = E^x f(X(\tau_D))$$

is harmonic in D . If $X(0) = x$ is close to ∂D , then the process hits ∂D very quickly with high probability, as can be seen by looking at the projection in the x direction of $X(0)$, which is a one dimensional Brownian motion. (See also Proposition 2 below.) Therefore, h_1 is continuous on the closure of D , and $h_1 = f$ on ∂D .

On the other hand, if a, b are chosen appropriately,

$$h_2(x) = \begin{cases} a \log \|x\| + b & \text{if } d = 2, \\ \frac{a}{\|x\|^{d-2}} + b & \text{if } d \geq 3 \end{cases}$$

has the same properties. Therefore, by the corollary, $h_1 \equiv h_2$ on D . Specifically,

$$\begin{aligned} & P^x(X(t) \text{ hits } \partial B(0, r_2) \text{ before } \partial B(0, r_1)) \\ &= \begin{cases} \frac{\log \|x\| - \log r_1}{\log r_2 - \log r_1} & \text{if } d = 2, \\ (r_2/\|x\|)^{d-2} \frac{\|x\|^{d-2} - r_1^{d-2}}{r_2^{d-2} - r_1^{d-2}} & \text{if } d \geq 3. \end{cases} \end{aligned}$$

Letting σ_r be the hitting time of $\partial B(0, r)$, we see that in both cases,

$$\lim_{r_1 \rightarrow 0} P^x(\sigma_{r_2} < \sigma_{r_1}) = 1, \quad 0 < \|x\| < r_2.$$

Since $\lim_{r \rightarrow 0} \sigma_r = \sigma_0$, the hitting time of 0, $P^x(\sigma_{r_2} < \sigma_0) = 1$, and since $\lim_{r \rightarrow \infty} \sigma_r = \infty$, it follows that $P^x(\sigma_0 = \infty) = 1$. Therefore, unlike the one dimensional case, Brownian motion does not hit points for $d \geq 2$:

$$P^x(X(t) = y \text{ for some } t > 0) = 0$$

for all x, y . On the other hand,

$$P^x(\sigma_{r_1} = \infty) = \lim_{r_2 \rightarrow \infty} P^x(\sigma_{r_2} < \sigma_{r_1}) = \begin{cases} 0 & \text{if } d = 2, \\ \frac{\|x\|^{d-2} - r_1^{d-2}}{\|x\|^{d-2}} & \text{if } d \geq 3 \end{cases}$$

for $\|x\| > r_1$, so Brownian motion is neighborhood recurrent (i.e., it hits open sets eventually with probability 1) for $d = 2$, but neighborhood transient for $d \geq 3$.

By Blumenthal's 0-1 law, $P^x(\tau_D = 0) = 0$ or 1 for each x , since $\{\tau_D = 0\} \in \mathcal{F}_0$. (Recall that the filtration is right continuous.) This suggests the following definition. (Note that $P^x(\tau_D = 0) = 0$ on D and $= 1$ on D^c .)

Definition. $x \in \partial D$ is regular if $P^x(\tau_D = 0) = 1$ and irregular if $P^x(\tau_D = 0) = 0$.

Here is a sufficient condition for regularity.

Proposition 1. (Cone condition) *If there is a truncated cone contained in D^c with base at $z \in \partial D$, then z is regular.*

Proof. Let C be the full cone and C_0 be the truncated cone. Then $P^z(X(t) \in C)$ is strictly positive and independent of t . Therefore,

$$P^z(\tau_D \leq t) \geq P^z(X(t) \in C_0) = P^z(X(t) \in C) - P^z(X(t) \in C \setminus C_0).$$

Letting $t \downarrow 0$, we see that $P(\tau_D = 0) > 0$, and therefore $P^z(\tau_D = 0) = 1$ by the Blumenthal 0-1 law, and hence 0 is regular.

Example. For $d \geq 2$, let $D = \{x : 0 < \|x\| < 1\}$. Then 0 is irregular, while the other boundary points are regular.

Recall that a function g is lower semicontinuous if $g(z) \leq \liminf_{x \rightarrow z} g(x)$. An increasing limit of continuous functions is lower semicontinuous.

Proposition 2. *For $t > 0$, the function $x \rightarrow P^x(\tau_D \leq t)$ is lower semicontinuous on R^d .*

Proof. Take $0 < s < t$ and use the Markov property to write

$$P^x(X(u) \in D^c \text{ for some } u \in (s, t]) = E^x P^{X_s}(\tau_D \leq t-s) = \int p_s(x, y) P^y(\tau_D \leq t-s) dy.$$

The right side is continuous in x , and the left side $\uparrow P^x(\tau_D \leq t)$ as $s \downarrow 0$.

Here is a much more precise version of Proposition 1, and one that shows that the function in Proposition 2 need not be continuous.

Example – the thorn. Suppose h is continuous, $h(r)$ is positive for $r > 0$, $h(0) = 0$, and $h(r)/r \uparrow$ in r for small r . Consider the thorn

$$T = \{(x_1, \dots, x_d) \in R^d : x_1 \geq 0, x_2^2 + \dots + x_d^2 \leq h^2(x_1)\}.$$

Note that this is a cone if $h(r) = cr$. Then 0 is regular for $D = T^c$ iff

$$\begin{aligned} \int_0^1 \left(\frac{h(r)}{r} \right)^{d-3} \frac{dr}{r} &= \infty, & d > 3, \\ \int_0^1 \left| \log \frac{h(r)}{r} \right|^{-1} \frac{dr}{r} &= \infty, & d = 3. \end{aligned}$$

(See page 259 of Ito and McKean.) We will not prove this in general, but will do the case of a cone below.

Now let D be the domain obtained by deleting T from the interior of $B(0, 1)$, where h is chosen so that 0 is irregular for D . Then $P^0(\tau_D \leq t) < 1$, but $P^x(\tau_D \leq t) = 1$ for $x \in \partial D \setminus \{0\}$, so this function is not continuous at 0.

Next we have a sufficient condition for our proposed solution h to the Dirichlet problem to attain the correct boundary value at a point on the boundary of D .

Theorem 3. *If $z \in \partial D$ is regular and f is bounded and continuous at z , then*

$$\lim_{\substack{x \rightarrow z \\ x \in D}} h(x) = f(z).$$

Proof. By Proposition 2, if $\epsilon > 0$, then

$$(4) \quad \liminf_{x \rightarrow z} P^x(\tau_D \leq \epsilon) \geq P^z(\tau_D \leq \epsilon) = 1.$$

Let σ_r be the hitting time of $\partial B(z, r)$. Then

$$\begin{aligned} E^x(|f(X(\tau_D)) - f(z)|, \tau_D < \infty) &\leq P^x(\tau_D \leq \sigma_r) \sup_{\substack{|y-z| \leq r \\ y \in \partial D}} |f(y) - f(z)| \\ &\quad + 2\|f\|_\infty P^x(\sigma_r \leq \tau_D < \infty). \end{aligned}$$

Pass to the limit using (4) to get

$$\lim_{\substack{x \rightarrow z \\ x \in D}} |h(x) - f(z)| = \lim_{\substack{x \rightarrow z \\ x \in D}} |h(x) - f(z)P^x(\tau_D < \infty)| \leq \sup_{\substack{|y-z| \leq r \\ y \in \partial D}} |f(y) - f(z)|.$$

Now let $r \downarrow 0$ to complete the proof.

Combining Theorems 1 and 3 and the corollary to Theorem 2, we have the following result:

Theorem 4. *If D is bounded, every point on ∂D is regular, and f is continuous on ∂D , then the unique solution to the Dirichlet problem is given by*

$$h(x) = E^x f(X(\tau_D)).$$

Remark. This result is often useful even if not all the hypotheses are satisfied. For example, let D be the punctured unit disk. As observed earlier, the origin is not a regular point. However, all boundary points for $D_\epsilon = \{x : \epsilon < |x| < 1\}$ are regular. If h is a solution to the Dirichlet problem on D , its restriction to D_ϵ is a solution to the Dirichlet problem on D_ϵ . Even though we don't know what the boundary values are on the inner boundary, Theorem 4 does imply that

$$h(x) = E^x f(X(\tau_{D_\epsilon})), \quad x \in D_\epsilon.$$

Since Brownian motion does not hit points in two dimensions, one can take the limit as $\epsilon \downarrow 0$ to conclude that

$$h(x) = E^x f(X(\tau_D)), \quad x \in D.$$

Note that this implies that boundary values cannot be assigned arbitrarily at irregular points on the boundary. Similar arguments allow one to handle the case of boundary functions that fail to be continuous at a few points.

From now on, we assume that all boundary points are regular, and investigate the Dirichlet problem when D is not bounded. Define g by

$$g(x) = P^x(\tau_D = \infty).$$

Theorem 5. (a) g is harmonic in D .

(b) For $z \in \partial D$, $\lim_{x \rightarrow z} g(x) = 0$.

(c) If $g \neq 0$, then there is no uniqueness for the Dirichlet problem on D .

Proof. (a) The proof is the same as in Theorem 1, based on the strong Markov property. (b) This follows from the regularity of z , since $\lim_{x \rightarrow z, x \in D} P^x(\tau_D \leq \epsilon) = 1$. (c) If h is any solution to the Dirichlet problem with boundary values f , then $h + cg$ is another solution for any constant c .

To apply Theorem 5, we need a criterion for $g \equiv 0$. It is provided by the concept of recurrence.

Definition. A Borel set A is said to be recurrent if for every $x \in R^d$,

$$P^x(\forall t > 0 \exists s > t : X(s) \in A) = 1.$$

If A is not recurrent, it is transient.

Theorem 6. $g \equiv 0$ on D iff D^c is recurrent.

Proof. If D^c is recurrent, then $P^x(\tau_D < \infty) = 1$ for all x , so $g \equiv 0$ on D . For the converse, suppose that $g \equiv 0$ on D . Then $P^x(\tau_D < \infty) = 1$ for every $x \in D$. If $x \in \partial D$, $P^x(\tau_D = 0) = 1$, since x is regular. On the other hand, $P^x(\tau_D = 0) = 1$ for every $x \in \bar{D}^c$ by path continuity. Therefore $P^x(\tau_D < \infty) = 1$ for all $x \in R^d$. By the Markov property,

$$P^x(X(s) \in D^c \text{ for some } s > t \mid \mathcal{F}_t) = E^{X(t)}(X(s) \in D^c \text{ for some } s > 0) = 1.$$

Therefore $P^x(X(s) \in D^c \text{ for some } s > t) = 1$ for all x, t , and so D^c is recurrent.

Examples. If $d = 2$ and D^c contains a ball, then $g \equiv 0$ on D . If $d \geq 3$ and $D = K^c$ for some compact set K , then $g \neq 0$ on D .

Here is the main result on the Dirichlet problem on unbounded domains.

Theorem 7. If f is bounded and continuous on ∂D and all points on ∂D are regular, then all bounded solutions h to the Dirichlet problem are given by

$$(5) \quad h(x) = E^x[f(X(\tau_D), \tau_D < \infty)] + cP^x(\tau_D = \infty),$$

for constant c .

Proof. Every function of the form in (5) is a solution to the Dirichlet problem with boundary values f . So, suppose h is any solution. We need to show that there is a constant c so that (5) holds. Take bounded domains D_n with all boundary points regular such that $D_n \uparrow D$. Then

$$(6) \quad h(x) = E^x h(X_{\tau_{D_n}}), \quad x \in D_n$$

by the uniqueness for bounded domains, since both sides of (6) are harmonic in D_n and have values h on ∂D_n . By the strong Markov property,

$$E^x[h(X(\tau_{D_{n+1}})) \mid \mathcal{F}_{\tau_{D_n}}] = E^{X(\tau_{D_n})}h(X(\tau_{D_{n+1}})) = h(X(\tau_{D_n})),$$

so that $h(X_{\tau_{D_n}})$ is a bounded martingale. By the martingale convergence theorem,

$$Z = \lim_{n \rightarrow \infty} h(X_{\tau_{D_n}})$$

exists a.s. and in L_1 . By path continuity,

$$X_{\tau_{D_n}} \rightarrow X_{\tau_D} \quad \text{a.s. on } \{\tau_D < \infty\}.$$

Since h has boundary values f on ∂D , it follows that

$$h(X_{\tau_{D_n}}) \rightarrow h(X_{\tau_D}) \quad \text{a.s. on } \{\tau_D < \infty\}.$$

By (6),

$$h(x) = E^x[f(X(\tau_D), \tau_D < \infty)] + E^x(Z, \tau_D = \infty)$$

To complete the proof, we need to show that there is a constant c independent of x so that $Z = c$ a.s. P^x on $\{\tau_D = \infty\}$. In doing so, we may assume $g(x) > 0$, since otherwise there is nothing to prove. We will carry out the proof in several steps:

1. Every bounded harmonic function h on R^d is constant. (Recall that we proved this earlier for random walks using martingales. Here we will prove it via coupling.) Couple two copies $X(t), Y(t)$ in such a way that each is a Brownian motion, $X(0) = x, Y(0) = y$, and $X(t) = Y(t)$ eventually a.s. To do so, it is enough to let the i th coordinates of the two processes move independently until they hit (which they will since until that time, $X_i(t) - Y_i(t)$ is a one dimensional Brownian, which must hit 0 eventually), and then let them move together. Then

$$|h(x) - h(y)| = |Eh(X(t)) - Eh(Y(t))| \leq 2\|h\|_\infty P(X(t) \neq Y(t)) \rightarrow 0$$

as $t \rightarrow \infty$.

2. Let $A = \{X(t_n) \in D^c \text{ for some sequence } t_n \uparrow \infty\}$. By the strong Markov property, $P^x(A)$ is harmonic on R^d , so there is a constant α so that $P^x(A) = \alpha$ on R^d . Therefore,

$$\alpha = P^{X(t)}(A) = P(A | \mathcal{F}_t) \rightarrow 1_A \quad \text{a.s.},$$

where the second equality comes from the Markov property, and the convergence from the martingale convergence theorem. Therefore $\alpha = 0$ or 1. (Of course, A is a tail event, so this is just the 0-1 law for tail events. This gives a different approach to this result.) Since we assumed $g(x) > 0$, $\alpha = 0$. Therefore, $X(t) \in D$ eventually a.s.

3. $h(X(\tau_D \wedge t))$ is a bounded martingale, so $\lim_{t \rightarrow \infty} h(X(\tau_D \wedge t))$ exists a.s., and therefore

$$(7) \quad L = \lim_{t \rightarrow \infty} h(X(t))$$

exists a.s. on $\{\tau_D = \infty\}$. By applying the Markov property at a large time, we see that (7) holds a.s. with no restriction. Here is the argument:

$$P^x(X(s) \in D \forall s > t) = E^x P^{X(t)}(\tau_D = \infty) \leq E^x P^{X(t)}(L \text{ exists}) = P^x(L \text{ exists}).$$

The left side tends to 1 as $t \rightarrow \infty$ by step 2, so the limit in (7) exists a.s.

4. By the 0-1 law, L is a constant $c(x)$ a.s. P^x for each x . This constant is harmonic, so does not depend on x by step 1 of the proof: $c(x) \equiv c$. Therefore, $Z = L = c$ on $\{\tau_D = \infty\}$ as required.