Due: Wednesday, May 23.

1. Problem 6.2 in Durrett.

2. Use Donsker’s Theorem to find the limiting distribution of \( \frac{1}{n} \max \{ k \leq n : S_k S_{k+1} \leq 0 \} \), where \( S_n \) are the partial sums of i.i.d. random variables with mean zero and finite variance.

3. Suppose \( f, f', f'' \) are continuous on \( \mathbb{R} \) and tend to zero at \( \pm \infty \), and let \( T(t) \) be the semigroup for standard one-dimensional Brownian motion:

\[
T(t)f(x) = \int_{-\infty}^{\infty} p_t(x, y)f(y)dy,
\]

where

\[
p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}.
\]

Show directly that

\[
\lim_{t \downarrow 0} \sup_x \left| \frac{T(t)f(x) - f(x)}{t} - \frac{1}{2} f''(x) \right| = 0,
\]

so that \( f \in \mathcal{D}(\mathcal{L}) \) and \( \mathcal{L}f = \frac{1}{2} f'' \). (Suggestion: Use scaling to write everything in terms of integrals with respect to the function \( p_t(0, \cdot) \).)

4. Suppose that the \( Q \)-matrix for a Markov chain on the countable set \( S \) satisfies \( q(x, y) \geq 0 \) for all \( y \neq x \) and \( \sum_y q(x, y) = 0 \) for all \( x \). Put \( c(x) = -q(x, x) \). Let \( p_t(x, y) \) be the minimal solution to the backward equation, and assume that it is stochastic and satisfies the Chapman-Kolmogorov equations (which is automatically true). Given a bounded function \( \phi \) on \( S \), prove that the unique bounded solution to

\[
u'(t, x) = \sum_z q(x, z)u(t, z), \quad u(0, x) = \phi(x)
\]

is given by

\[
u(t, x) = \sum_y p_t(x, y)\phi(y).
\]