(25) 1. Consider solving the equations

$$\begin{aligned} u^3 + xv - y &= 0 \\ v^3 + yu - x &= 0 \end{aligned} \tag{1}$$

for u, v in terms of x, y.

(a) Show that there are differentiable functions $u(x, y)$ and $v(x, y)$ defined in a neighborhood N of $(x, y) = (0, 1)$ so that $u(0, 1) = 1, v(0, 1) = -1$, and $u = u(x, y), v = v(x, y)$ solve (1).

Let $h(x, y, u, v) = (u^3 + xv - y, v^3 + yu - x)$. Then

$$h' = \begin{pmatrix} v & -1 & 3u^2 & x \\ -1 & u & y & 3v^2 \end{pmatrix}.$$

The assumption in the implicit function theorem is that the matrix

$$\begin{pmatrix} 3u^2 & x \\ y & 3v^2 \end{pmatrix}$$

be invertible at $(x, y) = (0, 1)$. At this point, the determinant of this matrix is 9, so the assumption is satisfied.

(b) Let $f(x, y) = (u(x, y), v(x, y))$ for $(x, y) \in N$. Compute $f'(x, y)$ in N.

Since $h((x, y), f(x, y)) = 0$ in N, the chain rule gives

$$\begin{pmatrix} v & -1 \\ -1 & u \end{pmatrix} + \begin{pmatrix} 3u^2 & x \\ y & 3v^2 \end{pmatrix} f'(x, y) = 0.$$

Solving gives

$$f'(x, y) = \frac{1}{9u^2v^2 - xy} \begin{pmatrix} -3v^3 - x & 3v^2 + ux \\ 3u^2 + vy & -3u^3 - y \end{pmatrix}.$$

(30) 2. In each case, decide whether the statement is true or false. If true, give a proof; if false, give a counterexample.

(a) If $f : X \to X$, where X is a metric space, satisfies $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$ for all $x, y \in X$, then there exists a point $x_0 \in X$ so that $f(x_0) = x_0$.

False. Let $X = (0, \infty)$ and $f(x) = \frac{1}{2}x$.

(b) If $f : R \to R$ satisfies $\{x : f(x) = c\}$ is measurable for each $c \in R$, then f is measurable.

False. Let $P \subset (0, 1)$ be a nonmeasurable set, and let

$$f(x) = \begin{cases} x & \text{if } x \in P; \\ -x & \text{if } x \in (0, 1) \backslash P; \\ 0 & \text{otherwise.} \end{cases}$$

(c) Suppose $f : B \to R^m$, where $B = \{x \in R^n : |x| < 1\}$ is differentiable in B and satisfies $||f'(x)|| \leq 5$ for all $x \in B$. Then $|f(x) - f(y)| \leq 5|x - y|$ for $x, y \in B$.

True. Let $g(t) = f(tx + (1-t)y)$ for $0 \leq t \leq 1$. Then $g'(t) = f'(tx + (1-t)y)(x - y)$, and therefore $|g'(t)| \leq 5|x - y|$. It follows that $|f(x) - f(y)| = |g(1) - g(0)| \leq f|x - y|$.

(25) 3. (a) Define the outer Lebesgue measure $m^*(A)$ of $A \subset R$.

$$m^*(A) = \inf\left\{ \sum l(I_n) : I_n \text{ are open intervals such that } A \subset \cup I_n \right\}.$$

(b) Prove directly from the definition that $m^*(A) = 0$ for any countable set A.

Write $A = \{a_1, a_2, ...\}$. Given $\epsilon > 0$, let $I_n = (a_n - \epsilon/2^n, a_n + \epsilon/2^n)$. Then $A \subset \cup I_n$ and $\sum l(I_n) = 2\epsilon$. Therefore $m^*(A) \leq 2\epsilon$ for every $\epsilon > 0$, so $m^*(A) = 0$.

(20) 4. (a) Suppose A_n is a sequence of measurable sets satisfying $\sum_{n=1}^{\infty} m(A_n) < \infty$. Show that $m(A) = 0$, where

$$A = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right)$$

is the set of points that are in infinitely many of the A_n's.

For each n,

$$A \subset \bigcup_{k=n}^{\infty} A_k,$$

so by monotonicity and subadditivity of m,

$$m(A) \leq m\left(\bigcup_{k=n}^{\infty} A_k \right) \leq \sum_{k=n}^{\infty} m(A_k).$$

Now let $n \to \infty$.

(b) Is this true under the weaker assumption that $m(A_n) \to 0$? Explain.

No. For a counterexample, let $A_1 = [0, 1], A_2 = [0, \frac{1}{2}], A_3 = [\frac{1}{2}, 1], A_4 = [0, \frac{1}{3}], A_5 = [\frac{1}{3}, \frac{2}{3}], A_6 = [\frac{2}{3}, 1], ...$