(15) 1. Prove directly (i.e., without using any theorems other than the completeness of R) that every sequence in $[0, 1]$ has a convergent subsequence.

(15) 2. (a) Define "$f : R \to R$ is continuous at x". (Do not use lim in your definition.)

(b) Prove directly from the definition that if f and g are both continuous at x, then so is $f + g$.

(15) 3. Suppose $\sum_n a_n$ converges, but does not converge absolutely. Show that the power series $\sum_n a_n x^n$ has radius of convergence 1.

(25) 4. Suppose that $f : X \to Y$ is continuous. In cases (a), (b), (d), (e), decide whether the statement is true or false. If true, prove it; if false give a counterexample.

(a) If X is compact, then $f(X)$ is compact.

(b) If Y is compact, then $f^{-1}(Y)$ is compact.

(c) Define: "X is connected".

(d) If X is connected, then $f(X)$ is connected.

(e) If Y is connected, then $f^{-1}(Y)$ is connected.

(30) 5. In each case, decide whether the statement is true or false. If true, give a proof. If false, give a counterexample.

(a) $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

(b) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = c$ implies $\lim_{n\to\infty} a_n = c$.

(c) If $f : (0, 1) \to R$ is continuous, then it is bounded.

(d) If f is uniformly continuous on $(0, 1)$, then $\lim_{n\to\infty} f(1/n)$ exists.

(e) The metric space Q with the usual metric is complete.

(f) The Cantor set contains both rational and irrational numbers.

(20) 6. (a) State the mean value theorem.

(b) Prove the mean value theorem. (Do not use the generalized mean value theorem.)

(c) Use the mean value theorem to prove Bernoulli's inequality: $(1 + x)^n \geq 1 + nx$ for $x > -1$ and positive integer n.

(20) 7. (a) State Taylor's Theorem.

(b) Suppose (i) f is twice differentiable, (ii) f'' is uniformly bounded, (iii) $f(0) = 0$, and (iv) $f'(0) \neq 0$. Show that there are positive numbers ϵ and C so that

$$\left| x - \frac{f(x)}{f'(x)} \right| \leq Cx^2 \quad \text{for all} \quad |x| \leq \epsilon.$$

(25) 8. Recall that x is a fixed point for a function f if $f(x) = x$. Consider the function

$$f(x) = \frac{x^3 + 1}{3}.$$

(a) Show that it has fixed points α, β, γ that satisfy

$$-2 < \alpha < -1, \quad 0 < \beta < 1, \quad 1 < \gamma < 2.$$

Now define a sequence x_n by $x_{n+1} = f(x_n)$.

(b) Show that if $x_1 > \gamma$, then $x_n \to \infty$ as $n \to \infty$.

(c) Show that if $\alpha < x_1 < \gamma$, then $x_n \to \beta$ as $n \to \infty$.

(15) 9. (a) Suppose $\sum_n |a_{n+1} - a_n| < \infty$. Prove that $\lim_{n\to\infty} a_n$ exists.

(b) Is the converse of (a) true? If so, prove it; if not give a counterexample.

(20) 10. Suppose that the plane has been written as a countable union $R^2 = \cup_{k=1}^{\infty} A_k$ of subsets A_k. Show that at least one A_k contains three points that are not colinear – i.e., that do not lie on a straight line.