1

1.1

Problem 1.) Let $A = \{2, 4, 6\}, \ B = \{4, 5, 6\}$. Then $(A \cup B)^c = \{1, 3\}$ and $(A \cap B)^c = \{1, 2, 3, 5\}.

Problem 2.)

\[ A^c = A^c \cap \Omega \]
\[ = A^c \cap (B \cup B^c) \]
\[ = (A^c \cap B) \cup (A^c \cap B^c). \]

Similarly,

\[ B^c = B^c \cap (A \cup A^c) \]
\[ = (A \cap B^c) \cup (A \cap B^c). \]

Using $A^c$ and $B^c$ from above, we see that

\[ (A \cap B) = A^c \cup B^c = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c). \]

Last, we have that $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$, so $(A \cap B)^c = \{2, 4, 5, 6\}$.

1.2

Problem 5.) 10%. Draw a Venn Diagram if this isn’t clear.

Problem 6.) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $P(1) = P(3) = P(5) = \frac{1}{5}$, $P(2) = P(4) = P(6) = \frac{2}{5}$.

Problem 7.) The sample space is any finite sequence with each element in the sequence in $\{1, 3\}$ except for the last number which is in $\{2, 4\}$. Also, the sample space contains all infinite sequences with all the elements in each sequence in $\{1, 3\}$. See that these sequences could be mapped directly to the binary numbers, meaning that the sample space is uncountable.

Problem 8.) Letting $a, b, c$ be the probabilities of winning against the 1st, 2nd, 3rd opponents respectively, we have that

\[ \text{Pr(Winning at least 2 games)} = (1 - a)bc + ab(1 - c) + abc = ab + bc - abc. \]

From here, it is clear that we want the second opponent to be the weakest so $b$ is the largest.

Problem 9.) It suffices to show that $A \cap S_i$ is disjoint from $A \cap S_j$, and that the union of $A \cap S_i$ over all $i$ is equal to $A$. Then its clear from the properties of a probability measure. For (b), use the partition \{(B \cap C), (B \cap C'\}, \{(B' \cap C), (B' \cap C')\}. Then

\[ P(A) = P(A \cap B \cap C) + P(A \cap B \cap C') + P(A \cap B \cap C') + P(A \cap B \cap C') \]

From part (a)

\[ = 2P(A \cap B \cap C) + P(A \cap B \cap C') + P(A \cap B \cap C') - P(A \cap B \cap C) \]

\[ = P(A \cap B) + P(A \cap C) + P(A \cap C') - P(A \cap B \cap C), \]

because $P(A \cap B) = P(A \cap B \cap C) + P(A \cap B \cap C')$, and a similar equation for $P(A \cap C)$.

A perhaps better alternative is to write

\[ P(A) = P(A \cap (B \cup C)) + P(A \cap (B \cup C')) = P(A \cap B) + P(A \cap C) + P(A \cap B' \cap C') - P(A \cap B \cap C) \]

Problem 10.)

\[ P((A \cap B) \cup (A^c \cap B)) = P(A \cap B^c) + P(A^c \cap B) \]
\[ = (P(A \cap B^c) + P(A \cap B)) + (P(A^c \cap B) + P(A \cap B)) - 2P(A \cap B) \]
\[ = P(A) + P(B) - 2P(A \cap B). \]
1.3

Problem 14.) (a) = \frac{1}{6}, (b) = \frac{1}{7}, (c) = \frac{11}{36}, (d) = \frac{1}{3} = \frac{10}{30}.

Problem 15.) A coin is tossed twice with probability \( p \) of landing on heads. Let \( X = " \)Two heads are tossed"\, 
\( Y = " \)First toss is a head"\,, and \( Z = " \)At least one toss is a head\". The question is, is it true that 
\[
\Pr(X|Y) \geq \Pr(X|Z)?
\]

Using Bayes rule, we can write out,
\[
\Pr(X|Y) = \frac{\Pr(Y|X) \Pr(X)}{\Pr(Y)} = \frac{1 \cdot p^2}{p} = p
\]
\[
\Pr(X|Z) = \frac{\Pr(Z|X) \Pr(X)}{\Pr(Z)} = \frac{1 \cdot p^2}{1 - (1 - p)^2}.
\]

From this calculation plus some algebra not shown, we see that it is always true, with equality only when 
\( p = 0 \).

Problem 16.) Using Bayes Rule, and letting \( A, B, C \) be the events that the first, second or third coin are 
picked, respectively, and \( H \) indicate the event that a head was flipped, we would like to calculate,
\[
\Pr(C|H) = \frac{\Pr(H|C) \Pr(C)}{\Pr(H)}
\]
\[
= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\Pr(H|A) \Pr(A) + \cdots + \Pr(H|C) \Pr(C)}
\]
\[
= \frac{\frac{1}{6}}{0 + \frac{1}{2} \cdot \frac{1}{2}} = \frac{1}{3}.
\]

Here, I’ve assumed that we pick between \( A, B, C \) uniformly at random.

Problem 17.) I will assume here that if one or more selected products are defective, the batch gets rejected. 
This problem is simple if one employs the hypergeometric pmf (see Main_Theorems.pdf Chapter 2). The probability that the batch passes is 1 minus the probability that it fails. The probability that it fails is given by
\[
\Pr(\text{Fail}) = \sum_{i=1}^{4} \frac{\binom{5}{i} \binom{95}{4-i}}{\binom{100}{4}}.
\]

To explain, the probability that one defective item gets chosen is the probability "choosing" 3 out of 95 good 
items and 1 out of 5 bad items, divided by all the ways one can pick 4 items out of 100, i.e.
\[
\frac{\binom{5}{1} \binom{95}{3}}{\binom{100}{4}}.
\]

Do this for 2, 3 and 4 bad items, and you’re done.

Problem 18.) Observe,
\[
\Pr(A \cap B|B) = \frac{\Pr(A \cap B \cap B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)} = \Pr(A|B).
\]

1.4

Problem 19.) Let \( A \) be the event that Alice does not find the paper in drawer \( i \). Then \( P(A^c) = p_id_i \), 
and \( P(A) = 1 - p_id_i \). Letting \( B \) be the event that the paper is in drawer \( j \), if \( i \neq j \), then \( A \cap B = B \), so
\[
P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} = \frac{p_j}{1 - p_id_i}.
\]

If \( i = j \), then using Bayes Rule,
\[
P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)P(A|B)}{P(A)} = \frac{p_i(1 - d_i)}{1 - p_id_i}.
\]
Problem 22.) Letting $p_i$ be the probability that a white ball is chosen from the $i$th jar, we have that

$$p_{i+1} = \frac{(m+1)p_i + m(1-p_i)}{m + n + 1}$$

and $p_1 = \frac{m}{m+n}$. Then $p_2$ is $\frac{m}{m+n}$ by directly substituting $p_1$ in the equation above. Since we have proved the base case and we have a recursive formula, the problem is solved.

Problem 23.) We can realize the original arrangement in 3 ways:

1. We could return to the original arrangement on the second swap, and on the fourth swap. For us to return on the second swap, we need to pick 1 out of $n$ balls from the first jar, and 1 out of $n$ balls from the second jar. We have a probability of $\frac{1}{n^2}$ of doing this. Then to do this process again and return on the fourth swap, the likelihood of doing so is $\frac{1}{n^4}$.

2. Another alternative is that on the second swap, we haven’t returned any balls to their original jar. The probability that this happens is $(1 - \frac{1}{n})(1 - \frac{1}{n})$. The only way to get back to the original arrangement now is to pick one of the correct balls on each of the subsequent swaps. This has probability $\frac{2}{n} \cdot \frac{2}{n}$ times $\frac{1}{n} \cdot \frac{1}{n}$. Thus, the probability that this situation occurs is $\frac{4}{n^4} (1 - \frac{1}{n})^2$.

3. Last, say on swap 2, we fix the arrangement of one jar, but not the other. We do the same on the third swap and fix it on the last swap, giving us a probability of $\frac{4}{n^4} (1 - \frac{1}{n})^2$.

Thus, the probability of returning to the original arrangement is

$$\frac{1}{n^4} + \frac{4}{n^4} (1 - \frac{1}{n})^2 + \frac{4}{n^4} (1 - \frac{1}{n})^2.$$