1. Announcements and Overview

2. Review of material

3. Examples

1. **Announcements and Overview:** One thing some people did not get on the previous homework was that \( \text{var}(X_1 + X_2 + \cdots + X_{10}) = 10 \text{var}(X_1) \), which is not the same as saying \( Y = 10X \), and \( \text{var}(Y) = 100 \text{var}(X) \).

2. **Important things from chapter 3:**

   (a) \( P(X \in B) = \int_B f_X(x) \, dx \)
   
   (b) \( E[X] = \int_{\mathbb{R}} x f_X(x) \, dx \)
   
   (c) \( \text{var}(X) = E[X^2] - (E[X])^2 \)
   
   (d) \( \frac{df_X}{dx} = f_X(x) \) when possible.
   
   (e) \( f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y) \, dy \)
   
   (f) \( P(X \in B | X \in A) = \frac{\int_A f_X(x) \, dx}{P(X \in A)} \)
   
   (g) Independence gives us \( f_{X,Y}(x,y) = f_X(x)f_Y(y) \), and this implies their expectation values multiply.

3. **Examples:** Mixed Random Variables: Let \( Y \) be a discrete random variable and \( Z \) be a continuous random variable. Finally, \( X \) is a random variable that follows the probability law of \( Y \) with probability \( p \) and the probability law of \( Z \) with probability \( 1 - p \). Then, \( F_X(x) = P(X \leq x) = pP(Y \leq x) + (1 - p)P(Z \leq x) = pF_Y(x) + (1 - p)F_Z(x) \). The expected value is defined as,

\[
\]

Consider the following example: The taxi stand and the bus stop near Al’s home are in the same location. Al goes there at a given time and if a taxi is waiting (this happens with probability \( \frac{2}{3} \)) he boards it. Otherwise he waits for a taxi or a bus to come, whichever comes first. The next taxi will arrive in a time that is uniformly distributed between 0 and 10 minutes, while the next bus will arrive in exactly 5 minutes. Find the CDF and the expected value of Al’s waiting time.

Letting \( A \) be the event that Al will find a taxi waiting or will be picked up by the bus after 5 minutes. Note that the probability of boarding the next bus, given that he has to wait in the first place, is 0.5. Al’s waiting time, call it \( T \), is a mixed random variable. With probability

\[
P(A) = \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{6},
\]

it is equal to its discrete component \( Y \), which corresponds to either finding a taxi waiting or boarding the bus at \( T = 5 \). This has pmf,

\[
p_Y(y) = \begin{cases} 
  \frac{2}{3P(A)} & \text{if } y = 0, \\
  \frac{1}{6P(A)} & \text{if } y = 5. 
\end{cases}
\]

This follows from \( p_Y(0) = P(Y = 0 | A) = \frac{2}{3P(A)} \). The complementary probability \( 1 - P(A) \) is the probability that the waiting time is equal to its continuous component \( Z \). This has pdf,

\[
f_Z(z) = \frac{1}{5}
\]
if $0 \leq z \leq 5$. Therefore, the CDF is given by

$$F_X(x) = P(A)F_Y(x) + (1 - P(A))F_Z(x),$$

or

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{5}{6} \cdot \frac{12}{15} + \frac{1}{6} \cdot \frac{z}{5} & 0 \leq x < 5 \\ 1 & 5 \leq x. \end{cases}$$

The expectation value, I’ll leave it to exercise, is $\frac{15}{12}$.

The new problem: A man is driving from Boston to New York area, a total distance of 180 miles. His average speed is uniformly distributed between 30 and 60 miles per hour. What is the PDF of the duration of the trip?

Letting $X$ be the speed and $Y = g(X)$ be the trip duration, we see that $g(X) = \frac{180}{X}$. To find the CDF of $Y$, we must calculate,

$$P(Y \leq y) = P\left(\frac{180}{X} \leq y\right) = P\left(\frac{180}{y} \leq X\right).$$

We use the given uniform PDF of $X$, which is

$$f_X(x) = \frac{1}{30} \text{ if } 30 \leq x \leq 60,$$

and the corresponding CDF,

$$F_X(x) = \begin{cases} 0 & x \leq 30 \\ \frac{x - 30}{30} & 30 \leq x \leq 60 \\ 1 & 60 \leq x \end{cases}.$$ 

Thus,

$$F_Y(y) = P\left(\frac{180}{y} \leq X\right) = 1 - F_X\left(\frac{180}{y}\right)$$

and this equals

$$= \begin{cases} 0 & y \leq 3 \\ 2 - \frac{6}{y} & 3 \leq y \leq 6 \\ 1 & 6 \leq y \end{cases}.$$

Upon differentiation, we get that

$$f_Y(y) = \begin{cases} 0 & y \leq 3, \ 6 \leq y \\ \frac{6}{y^2} & 3 \leq y \leq 6 \end{cases}.$$

Notice that the expected trip duration is not 4.5 hours as one might expect, but it is actually less than that. This is one justification for why speeding usually doesn’t get you where you are going much faster. Another way to see it is this: say the normal driving speed is 60. To cut your time in half, you must go 120 (60 mph faster), but to double your time, you need to go 30 mph. This means that a speed up from 60 to 70 mph will have less of an effect on the total trip time than a speed up from 50 to 60 will. In short, the faster you are going, the less speeding will help you.

**Problem 1.)** The random variable $Y = g(x)$ has the pmf

$$p_Y(1) = P(X \leq 1/3) = \frac{1}{3}, \ p_Y(2) = \frac{2}{3}.$$

Thus,

$$E[Y] = \frac{1}{3} + \frac{2}{3} = \frac{5}{3}.$$
Problem 2.) Calculate:
\[
\int_{-\infty}^{\infty} f_X(x) \, dx = 2 \cdot \frac{1}{2} \int_{0}^{\infty} \lambda e^{-\lambda x} \, dx = \frac{1}{2}^2 = 1.
\]
(1)
By symmetry, we have that \( E[X] = 0 \), and
\[
E[X^2] = \int_{\mathbb{R}} x^2 \frac{\lambda}{2} e^{-\lambda x} \, dx = \frac{2}{\lambda^2}.
\]
Therefore, \( \text{var}(X) = \frac{2}{\lambda^2} \).

Problem 5.) Let \( A = \frac{bh}{2} \) be the area of the given triangle. From the randomly chosen point, draw a line parallel to the base, and let \( A_x \) be the area of the triangle formed. We have that
\[
A_x = b(h - x)^2/(2h), \quad \text{and for } x \in [0, h], \quad \text{we have}
\]
\[
F_X(x) = 1 - P(X > x) = 1 - \frac{A_x}{A} = 1 - \frac{b(h - x)^2/(2h)}{bh/2} = 1 - \left( \frac{h - x}{h} \right)^2
\]
Differentiating this gives the PDF.

Problem 6.) Let \( X \) be the waiting time and \( Y \) be the number of customers found. Then for \( x \geq 0 \),
\[
F_X(x) = \frac{1}{2} P(X \leq x | Y = 0) + \frac{1}{2} P(X \leq x | Y = 1).
\]
Since
\[
P(X \leq x | Y = 0) = 1,
\]
and
\[
P(X \leq x | Y = 1) = 1 - e^{-\lambda x},
\]
we obtain,
\[
F_X(x) = \frac{1}{2} (2 - e^{-\lambda x})
\]
when \( x \geq 0 \). Note that \( X \) is a mixed RV.

Problem 20.) The expected time can be conditioned on the events that the first student stays no more than 5 minutes, and on the first student staying longer than 5 minutes. In other words, letting \( T \) be the time of the entire appointments, and \( T_i \) be the length of the stay of the \( i \)-th student,
\[
E[T] = (5 + E[T_2]) P(T_1 \leq 5) + (E[T_1 | T_1 > 5] + E[T_2]) P(T_1 > 5).
\]
The problem gives us that \( E[T_2] = 30 \), and due to memorylessness,
\[
E[T_1 | T_1 > 5] = 5 + E[T_1] = 35.
\]
Also,
\[
P(T_1 \leq 5) = 1 - e^{-5/30},
\]
\[
P(T_1 > 5) = e^{-5/30}.
\]
Therefore,
\[
E[T] = 35(1 - e^{-5/30}) + 65e^{-5/30} \approx 60.
\]