1. Announcements and Overview: The fifth homework is due today in discussion!

2. Review of material from sections 2.3-2.5, and the previous homework: The only thing I’d like to talk about today is joint probability distribution functions. Expectation values and variance are discussed in 2.3 and 2.4. We denote the joint pmf of two random variables as

\[ p_{X,Y}(x, y) = \text{Pr}(X = x, Y = y) = \text{Pr}(\{X = x\} \cap \{Y = y\}) \].

If \( A \) is the set of all pair \((x, y)\) with a certain property, then

\[ \text{Pr}((X, Y) \in A) = \sum_{(x, y) \in A} p_{X,Y}(x, y) \].

Perhaps most importantly, we can compute the marginal distribution \( p_X \) (or similarly \( p_Y \)) by,

\[ p_X(x) = \sum_y p_{X,Y}(x, y) \].

Other bullet points might be that if \( Z = g(X, Y) \), then \( Z \) is also a random variable, and

\[ \mathbb{E}[Z] = \sum_x \sum_y g(x, y)p_{X,Y}(x, y) \].

3. Examples and previous homework:

(a) Problem 33.) This problem is fun to think about. Since we don’t know the bias, it would be hard to make an claims like "flip it x number of times, and if its heads more than m times, consider it as one decision." Instead, if we consider grouping coin flips into sets of two, then we have 4 possible outcomes, namely \((h, h); (h, t); (t, h); (t, t)\). We see that the probability of \((h, t)\) is the same as the probability of \((t, h)\). Therefore, if we throw away the occurrences of \((h, h)\) and \((t, t)\), we can decide on decision 1 if \((t, h)\) shows up first, and decision 2 if \((h, t)\) shows up first. This is a fair way to make the decision using a biased coin.

(b) Problem 35.) If one thinks of each piece as being a Bernoulli trial, then the probability of \( k \) pieces simultaneously working is \( \binom{n}{k} p^k(1-p)^{n-k} \). Thus, the probability that the machine is working at a given moment is,

\[ \text{Pr}(\text{Machine is functional}) = \sum_{i=k}^{n} \binom{n}{i} p^i(1-p)^{n-i} \].

(c) Problem 34.) The system only works if each of the three components work. Each component works with probability

\[ p_1 = p, p_2 = 1 - (1-p)(1-p(1-(1-p)^3)), p_3 = 1 - (1-p)^2 \].

Here, the only difficult one is \( p_2 \). We see that we need either the single circuit to be working or the more complicated subsubsystem. For the subsubsystem to be down, we need both a single failure and one of the three to fail.
(d) Problem 38.) Let $p_T$ be the probability that at least 6 out of the 8 remaining holes are won by Telis, and $p_W$ be the probability that Wendy wins 4 out of the 8. Using the binomial formula, we have

$$p_T = \sum_{k=6}^{8} \binom{8}{k} p^k (1-p)^{8-k}, \quad p_W = \sum_{k=4}^{8} \binom{8}{k} (1-p)^k p^{8-k}.$$ 

The amount of money that Telis should get is $10p_T/(p_T + p_W)$.

(e) Problem 40.) Let $A$ be the event that the first $n-1$ tosses produce an even number of heads, and let $E$ be the event that the $n$th toss is a head. We can obtain an even number of heads in $n$ tosses in two distinct ways:

1) There is an even number of heads in the first $n-1$ tosses, and the $n$th toss results in tails. This is the event $A \cap E^c$.
2) There is an odd number of heads in the first $n-1$ tosses, and the $n$th toss results in heads. This is the event $A^c \cap E$.

Using also the independence of $A, E$, we have that

$$q_n = P( (A \cap E^c) \cup (A^c \cap E) )$$

$$= P(A \cap E^c) + P(A^c \cap E)$$

$$= P(A)P(E^c) + P(A^c)P(E)$$

$$= (1-p)q_{n-1} + p(1-q_{n-1}).$$

Now we use induction to prove the equation works, and letting

$$q_{n-1} = \frac{1 + (1-2p)^{n-1}}{2},$$

we substitute this in to the expression for $q_n$ to establish the formula.

(f) Problem 56.) For part (a), there are $\binom{9}{3}$ ways to pick the 4 lower level classes and $\binom{10}{3}$ ways to choose the upper division classes. By the multiplication principle, we have that there are $\binom{9}{3}\binom{10}{3}$ different curriculum.

Part (b): If we do not choose $L_1$, then we must choose $L_2$ and $L_3$. Since we need 4 lower division courses, we must pick 2 more out of the remaining 5. Since we can only sample from the upper division courses $6-10$, we get that $\binom{5}{2}\binom{5}{3}$ curriculum are available. If we choose $L_1$ and not $L_2$ or $L_3$, then we have $\binom{5}{2}\binom{5}{3}$ curriculum. If we choose $L_1, L_2$, and $L_3$, we get that there are $5\binom{10}{3}$ curriculum. And finally, if we choose $L_1$ and one, but not both of $L_2, L_3$, then we have $2 \cdot \binom{5}{2}\binom{5}{3}$ choices. If we add up all the above possibilities (since they are all disjoint situations) we get that is the number of curriculum.

(g) The Jacket Problem: Suppose $n$ people all give their jackets to the careless employee to hang up. Being careless, the employee forgets to put identifying tags on each of them, and when the evening is over, hands them back at random. What is the expected value of $X$, the number of people who get their (correct) jacket back?

If we let $X_i$ be a new random variable that takes the values 0 and 1, 1 if the person gets their jacket back, then

$$E[X_i] = 1 \cdot \frac{1}{n} + 0 \cdot (1 - \frac{1}{n}) = \frac{1}{n}.$$ 

Since $X$ is a linear function of the $X_i$’s, we have $E[X] = n \cdot \frac{1}{n} = 1$.

(h) Yet another birthday problem: Suppose you go to a party with 100 guests total, what is the probability that exactly two of the guests have the same birthday as you?

Since we can treat each guest as a Bernoulli trial, with the probability of success (meaning the same birthday) being $\frac{1}{365}$, we can just write down,

$$\binom{99}{2} \frac{1}{365^2} \binom{364}{97} \frac{364}{365},$$

as the probability that exactly two people have the same birthday as you.
(i) **Problem 2.3.** Fisher and Spassky play a chess match in which the first player to win a game wins the match. After 10 successive draws, the match is declared a draw. Each game is won by Fischer with probability 0.4 and by Spassky with probability 0.3. Given that all games are independent, what is the probability that Fischer wins the match? What is the PMF for the duration of the match?

Letting \( L \) be the duration of the match, we have that if Fischer wins a match consisting of \( L \) games, then \( L - 1 \) draws must occur prior to the win. Summing over all possible lengths, we have that,

\[
\text{Pr}(\text{Fischer Wins}) = \sum_{l=1}^{10} (0.3)^{l-1}(0.4) = 0.571....
\]

First, we know that the probability of \( L \) not equal to an integer between 1 and 10 is zero. Also, if all ten games are played, this occurs with the probability \( (0.3)^9 \), because 9 draws would need to occur. In general, we know that the probability that the match will end on a given game is 0.7, and so the PMF is,

\[
p_L(l) = \begin{cases} 
(0.3)^{l-1}(0.7) & l = 1, 2, \ldots, 9 \\
(0.3)^9 & l = 10 \\
0 & \text{else} 
\end{cases}
\]

4. **Hand back homeworks. Any questions? Problems due this week: Chapter 2, Numbers 2; 4; 6; 8; 10; 14; 16; 21.**