1. Announcements and Overview

2. Review of material

3. Examples

4. Questions about homework and previous exams

1. **Announcements and Overview:** The third homework is due today in discussion! The sketch of the solutions to the second homework are posted.

**Review of Homework Problems for Test 1:**

1. **Problem 15.** A coin is tossed twice with probability $p$ of landing on heads. Let $X = \text{"Two heads are tossed"}$, $Y = \text{"First toss is a head"}$, and $Z = \text{"At least one toss is a head"}$. The question is, is it true that $\Pr(X|Y) \geq \Pr(X|Z)$?

   Using Bayes rule, we can write out,
   \[
   \Pr(X|Y) = \frac{\Pr(Y|X) \Pr(X)}{\Pr(Y)} = \frac{1 \cdot p^2}{p} = p \tag{1}
   \]
   \[
   \Pr(X|Z) = \frac{\Pr(Z|X) \Pr(X)}{\Pr(Z)} = \frac{1 \cdot p^2}{1 - (1 - p)^2}. \tag{2}
   \]

   Above, it is noted that $\Pr(Y|X) = 1$, $\Pr(X) = p^2$, and $\Pr(Z) = 1 - \Pr(Z^c)$. From this calculation plus some algebra not shown, we see that it is always true, with equality only when $p = 0, 1$.

2. **Problem 19.** Let $A$ be the event that Alice does not find the paper in drawer $i$. Then $P(A^c) = p_i d_i$, and $P(A) = 1 - p_i d_i$. Letting $B$ be the event that the paper is in drawer $j$, if $i \neq j$, then $A \cap B = B$, so
   \[
   P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} = \frac{p_j}{1 - p_i d_i}.
   \]
   If $i = j$, then using Bayes Rule,
   \[
   P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)P(A|B)}{P(A)} = \frac{p_i(1 - d_i)}{1 - p_i d_i}.
   \]

3. **Problem 22.** Letting $p_i$ be the probability that a white ball is chosen from the $i$ th jar, we have that
   \[
   p_{i+1} = \frac{(m + 1)p_i}{m + n + 1} + \frac{m(1 - p_i)}{m + n + 1},
   \]
   and $p_1 = \frac{m}{m + n}$. Then $p_2$ is $\frac{m}{m + n}$ by directly substituting $p_1$ in the equation above. Since we have proved the base case and we have a recursive formula, the problem is solved.

4. **Problem 23.** We can realize the original arrangement in 3 ways,

   (a) We could return to the original arrangement on the second swap, and on the fourth swap. For us to return on the second swap, we need to pick 1 out of $n$ balls from the first jar, and 1 out of $n$ balls from the second jar. We have a probability of $\frac{1}{m^2}$ of doing this. Then to do this process again and return on the fourth swap, the likelyhood of doing so is $\frac{1}{n^2}$.
(b) Another alternative is that on the second swap, we haven’t returned any balls to their original jar. The probability that this happens is \((1 - \frac{2}{n})(1 - \frac{1}{n})\). The only way to get back to the original arrangement now is to pick one of the correct balls on each of the subsequent swaps. This has probability \(\frac{2}{n} \cdot \frac{2}{n} \) times \(\frac{1}{n} \cdot \frac{1}{n}\). Thus, the probability that this situation occurs is \(\frac{4}{n^4}(1 - \frac{1}{n})^2\).

(c) Last, say on swap 2, we fix the arrangement of one jar, but not the other. We do the same on the third swap and fix it on the last swap, giving us a probability of \(\frac{4}{n^4}(1 - \frac{1}{n})^2\).

Thus, the probability of returning to the original arrangement is
\[
\frac{1}{n^4} + \frac{4}{n^4}(1 - \frac{1}{n})^2 + \frac{4}{n^4}(1 - \frac{1}{n})^2.
\]

5. **Problem 8.** Letting \(a, b, c\) be the probabilities of winning against the 1st, 2nd, 3rd opponents respectively, we have that
\[
\Pr(\text{Winning at least 2 games}) = (1 - a)bc + ab(1 - c) + abc = ab + bc - abc.
\]

From here, it is clear that we want the second opponent to be the weakest so \(b\) is the largest.

6. **Problem 30.** A quick comparison shows us that the strategy is equally as effective. If the dogs agree, the hunter picks the correct path with probability \(p^2\). If one of the dogs picks the right path and he follows it, this has probability \(2 \cdot \frac{1}{2} p(1 - p)\), since this can happen in two ways. Adding these disjoint events up we see that its equally as effective.

7. **Problem 31.**
(a) The probability according to the diagram is \(p(1 - \epsilon_0) + (1 - p)(1 - \epsilon_1)\).
(b) We multiply the probabilities together because of independence, so \((1 - \epsilon_0)(1 - \epsilon_1)^3\).
(c) It would be the probability of sending 0 correctly 3 times plus the probability of sending 0 correctly twice, meaning \((1 - \epsilon_0)^3 + \binom{3}{2} \epsilon_0(1 - \epsilon_0)\).
(d) Find \(\epsilon_0\) such that \((1 - \epsilon_0)^3 + \binom{3}{2} \epsilon_0(1 - \epsilon_0) > (1 - \epsilon_0)\). Omitting the algebra, \(\epsilon_0 \in [0, \frac{1}{2}]\).
(e) Given we recieve 101, the probability that it came from a 000 can be found by Bayes theorem.

The events in the following are intuitive.
\[
P(0|101) = \frac{P(101|0)P(0)}{P(101)} = \frac{pe_0^2(1 - \epsilon_0)}{pe_0^3(1 - \epsilon_0) + (1 - p)e_1(1 - \epsilon_1)^2}.
\]

Note that the total law of probability was used in the denominator.

8. **Problem 32.** It is important for this problem to note that it doesn’t specify if the sibling is older or younger than the king. Also, the assumptions I will make are that 1) the king is the first born male, and 2) that only males and females are born each with equal probability \(1/2\).

Define the relevant events, \(A = \text{"The king has a male sibling"}, \ B = \text{"The family has at least 1 boy}.\) Letting \(b\) and \(g\) represent each gender, we see that our sample space has size 4, \(\Omega = \{(g, g); (g, b); (b, g); (b, b)\}\). Then \(\Pr(B) = \frac{3}{4}\) and \(\Pr(A) = \frac{1}{2}\). Thus, using the definition of conditional probability,
\[
\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1}{3}.
\]

9. **Problem 33.** This problem is fun to think about. Since we don’t know the bias, it would be hard to make an claims like "flip it x number of times, and if its heads more than m times, consider it as one decision." Instead, if we consider grouping coin flips into sets of two, then we have 4 possible outcomes, namely \((h, h); (h, t); (t, h); (t, t)\). We see that the probability of \((h, t)\) is the same as the probability of \((t, h)\). Therefore, if we throw away the occurrences of \((h, h)\) and \((t, t)\), we can decide on decision 1 if \((t, h)\) shows up first, and decision 2 if \((h, t)\) shows up first. This is a fair way to make the decision using a biased coin.
10. Important to note: A probability model in this class is a defined technical term. In order to fully specify a probability model, one must explicitly define the sample space $\Omega$ (correctly) and the probability law from $\Omega$ to the interval $[0,1]$. Note that $\Omega$ only needs to contain distinct outcomes, meaning we wouldn’t write that $\Omega = \{1, 2, 2, 3, 4, 4, 5, 6, 6\}$ in the case where rolling an even number is twice as likely as an odd number. Also, recall the normal properties that a probability function must have, nonnegativity, subadditivity, and so on.

Other Examples:

1. Suppose we are in a game show, there are 3 curtains and a valuable prize is behind one curtain. The contestant picks one curtain, and before it is opened, the host opens a different curtain which reveals nothing. Do you keep the curtain ou picked or switch to the other unopened curtain?

The answer depends on whether or not the host knows which curtain has the prize, and assumes the host always opens another curtain regardless of where the prize is. Let $A_j$ be the event that the prize is behind curtain $j$ and let $H_k$ be the event that the host opens curtain $k$ and reveals no prize. Let calculate $\Pr(A_i|H_k)$, the probability that we picked the correct curtain on the first pick. Clearly,

$$\Pr(A_j|H_k) = 1 - \Pr(A_i|H_k).$$

Also,

$$\Pr(A_i|H_k) = \frac{\Pr(H_k|A_i) \Pr(A_i)}{\Pr(H_k)},$$

and $\Pr(A_i) = \frac{1}{3}$ by classical probability calculation. $\Pr(H_k|A_i)$ depends on the model for whether or not the host is aware of where the prize is. Let’s take the model where he does know where it is. Then he chooses with complete randomness between the two curtains without the prize, $\Pr(H_k|A_i) = \frac{1}{2}$. To evaluate $\Pr(H_k)$, use the law of total probability,

$$\Pr(H_k) = \Pr(H_k|A_i) \Pr(A_i) + \Pr(H_k|A_j) \Pr(A_j) + \Pr(H_k|A_k) \Pr(A_k) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 + 0 \cdot \frac{1}{3} = \frac{1}{2}. $$

Thus,

$$\Pr(A_i|H_k) = \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3},$$

which is the probability that it is good to stay.

Let’s say that the host is clueless about which curtain has the prize, and randomly opens another curtain other than the one that the contestant picked. Now we have that

$$\Pr(H_k|A_i) = \frac{1}{2}$$

because there are two unchosen curtains to select, and they’re guaranteed to not have the prize due to $A_i$. Also, $\Pr(H_k|A_j) = \frac{1}{2}$ since the host is clueless, and then

$$\Pr(A_i|H_k) = \frac{\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{2}. $$

Therefore, if the host is clueless about the location of the prize, it doesn’t matter if you switch.

4th Challenge Question of the month\(^2\): The FBI receives a tip that 4 members of a crime group are working in a particular office building. By the time the FBI investigators show up, 40 people from the office building are still there, but the crime members may have escaped. The agent estimates that each criminal has a 5\% chance of having escaped, independently of each other.

\(^1\)I won’t do this in discussion.

\(^2\)The challenge problems this quarter come from my first probability class with Peter Kramer at RPI, who gets the credit for devising these questions.
Find the probability that at least one criminal escaped.

Later, after interviewing 15 people, the agent found 1 criminal among the 15 (assume the interviewing process is perfect). Derive a revised formula for the probability that at least one criminal escaped, in light of the new information. (Think Bayesian).