1. Announcements and Overview

1. Announcements and Overview: The second homework is due today in discussion! The sketch of the solutions to the first homework are posted. Begin new to TAing probability, I have updated my course policy as listed below:

(a) I will not go over current homework problems in discussion. This is an unfair treatment of the material because some students have not handed their homeworks in, and are able to just copy what is written directly off the board. However, the discussion following each homework I will go over approximately half of the homework problems.

(b) I will accept homework at the beginning OR the end of discussion. However, omitting the possibility of an emergency, homeworks handed in mid discussion won’t be accepted. This policy is due to students walking up the front of the room during discussion, handing in their homework, and leaving. It is both disrupting and disrespectful to the TA and the students in class, which brings me to the next point.

(c) Except in the case of an emergency or the bathroom, if you plan to attend discussion and then leave, do so in the first 5 minutes. See the last sentence above. Be respectful to your peers.

(d) I will be grading half of the homework problems at random, and presenting the other half the following week. Each homework is out of 100, and we do not accept homeworks after 3:50pm on the Thursday it is due.

(e) If there are any questions about the new policies, please either send me an email or see me in office hours.

2. Review of Material, Conditional Probability and Total Probability:

(a) We use the notation \( \Pr(A|B) \) to mean that we know event \( B \) occurs, and given this restriction, \( \Pr(A|B) \) is the probability that \( A \) has occurred too. Mathematically, this is equivalent to

\[
\Pr(A|B) := \frac{\Pr(A \cap B)}{\Pr(B)}
\]

Here we have the assumption that \( \Pr(B) \geq 0 \).

(b) What we have done effectively is we have shrunk our sample space from all of \( \Omega \) down to the set \( B \), and then renormalized. This means that we still have the familiar probability laws, expect on a restricted space of possibilities. For example,

\[
\Pr(A \cup B|C) \leq \Pr(A|C) + \Pr(B|C),
\]

with equality if \( A \) and \( B \) are disjoint.

(c) If all the outcomes are equally likely, we have that \( \Pr(A|B) = \frac{\text{number of elements in } A \cap B}{\text{number of elements in } B} \). For example, say we flip a coin and the first toss is a tail. We want to find the probability that more heads than tails will occur. Then we are restricted to the sample space \( \{T HH\}, \{T HT\}, \{TT H\}, \{TT T\} \). We count the number of outcomes consistent with more heads than tails, and we see that the probability of this is \( \frac{1}{4} \), given that a tail is flipped first.

(d) One less familiar rule is the multiplication rule,

\[
P(\cap_{i=1}^{n} A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|\cap_{i=1}^{n-1} A_i).
\]

Using the definition of \( \Pr(A|B) \), we can directly verify this relationship.
(e) The law of total probability: Let $A_1, \ldots, A_n$ be disjoint events that partition the sample space. Then for any event $B$, we have that

$$P(B) = P(A_1 \cap B) + \cdots + P(A_n \cap B) = P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n).$$

(f) Inference and Bayes’ Rule: For events $A$ and $B$, we have

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}.$$ 

We can substitute in the law of total probability into the denominator to simplify many calculations. This rule is often used for inference, or given an “effect”, we would like to figure out what the probability that a particular “cause” is what occurred. We say that $P(A|B)$ is the posterior probability of event $A$, and $P(A)$ is the prior probability. This means that given information, that $B$ happened, we look to adjust our previous beliefs $P(A)$, and we do this using Bayes formula.

3. Examples:

(a) Radar detection problem: Say that if an unknown aircraft is in a restricted area, the radar detects it with probability 0.99. If an aircraft is not present, the radar generates a false alarm with probability 0.1. We assume that a foreign aircraft is present with probability 0.05. What is the probability that an aircraft is present given that an alarm sounds?

Letting $A$ be the event that an aircraft is present and $B$ be the event that the radar sounds an alarm, we wish to find $Pr(A|B)$. This is equal to $Pr(A)Pr(B|A) = 0.05 \cdot (1 - 0.99) = 0.0005$.

(b) This example demonstrates the multiplication rule. Three cards are drawn from an ordinary 52 card deck without replacement. We want to know the probability that none of the cards is a heart. A cumbersome approach might be to count all the ways 3 cards can be selected out of a population of 52 without replacement such that none of the cards are a heart, or, we could use the multiplication rule. Let $A_i$ be the event that the $i$-th card is not a heart.

To find $P(A_1 \cap A_2 \cap A_3)$, we can compute,

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2).$$

We have that $P(A_1) = \frac{39}{52}$. Given that this is true, we have that $P(A_2|A_1) = \frac{38}{51}$. Last, we have $P(A_3|A_1 \cap A_2) = \frac{37}{50}$. Thus,

$$P(A_1 \cap A_2 \cap A_3) = \frac{39 \cdot 38 \cdot 57}{52 \cdot 51 \cdot 50}.$$

(c) This example goes over the total probability law. In a tournament, you have a 0.3 chance at winning against half the players (type 1 players), 0.4 chance at beating a quarter of the players (type 2), and 0.5 chance at beating the remaining players (type 3). You play a game against a random opponent. What is the probability of winning?

Letting $A_i$ be the event of playing an opponent of type $i$, we have,

$$P(\text{Winning } := B) = P(A_1)P(B|A_1) + \cdots + P(A_3)P(B|A_3),$$

or

$$P(B) = 0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5 = 0.375.$$

(d) Letting $A$ be the event that an aircraft is present and $B$ is the event that the radar generates an alarm, we are given that $P(A) = 0.05$, $P(B|A) = 0.99$, and $P(B|A^c) = 0.1$, then we can calculate the probability that an aircraft is present given that an alarm sounds:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$= \frac{0.05 \cdot 0.99}{0.05 \cdot 0.99 + 0.95 \cdot 0.1} = 0.3426.$$
(e) Returning to the first example, letting $B$ be the event of winning, we can calculate the probability that we played an opponent of type 1. Just write,

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(A_1)P(B|A_1) + \cdots + P(A_3)P(B|A_3)}$$  
$$= \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5} = 0.4. \quad (6)$$

(f) Another important example of Bayes’ rule, the probability of a False Positive. A test for a certain rare disease is assumed to be correct 95% of the time, meaning if the person has the disease, it’ll test positive 95% of the time. If the person doesn’t, it’ll draw a negative 95% of the time too. The probability of having the disease within the given population is 0.001. Given that a positive test occurred, what is the probability that the test is correct?

Letting $A$ be the event that the person has the disease, and $B$ is the event that the test results are positive, we calculate,

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$  
$$= \frac{0.001 \cdot 0.95}{0.001 \cdot 0.95 + 0.999 \cdot 0.05} = 0.0187. \quad (8)$$

Even through the reliability seems good, due to the fact that the disease is rare, it is likely that the test drew a false positive.

(g) From last time, the first birthday problem: Say there is a group of $m$ individuals including myself. Assuming that the distribution of births is uniform across the year, what is the probability that someone else in the group has my birthday? First, every individual “chooses” a birthday uniformly out of a list of 365 possible birthdays. Thus, we can set up the sample space consisting of $m-1$ selections from a population of 365 (since the first selection is fixed on my birthday). These selections are with replacement and are ordered. Therefore, there are $365^{m-1}$ possible ordered sequences of birthdays for the group. Now, if we can calculate the event that no one has the same birthday as me, the probability we seek will be 1 minus the probability no one has my birthday. The number of sequences consistent with this event are $364^{m-1}$, which is $m-1$ individuals selecting out of a population of 364 birthdays that don’t coincide with mine. Therefore, the probability that someone has my birthday is $1 - \left(\frac{364}{365}\right)^{m-1}$. If $m \geq 253$, the probability is greater than 0.5.

(h) Now, how about a second birthday problem. Say we are just interested in $A$ = “Any two or more people have the same birthday”. Again, $m$ people choose their birthdays (in an ordered manner) from a population of 365, meaning $|\Omega| = 365^m$. Now, $A^c$ = no two people have the same birthday, is probably the easier event to calculate. The number of outcomes corresponding to this event is the number of $m$ ordered samples without replacement from a population of 365. This is $|A^c| = \frac{365!}{(365-m)!}$, if $m < 365$. Therefore,

$$P(A) = 1 - \frac{365!}{365^m(365-m)!}.$$

If $m \geq 23$, then the probability of this event is greater than 0.5.

3rd Challenge Question of the month\textsuperscript{1}:
Suppose a company with 40 employees wishes to interview 8 of them at random to determine their happiness with working conditions. Suppose 15 are happy, 13 are indifferent, and 12 are disgruntled. Set up a probability model to assess the probability that at least one person from each opinion group will be interviewed.

Too easy? Suppose that 1 of the disgruntled employees is about to quit. Calculate the probability that the interview will include at least 2 happy, 2 indifferent, and 2 disgruntled employees, one of which is the one who is about to quit.

\textsuperscript{1}The challenge problems this quarter come from my first probability class with Peter Kramer at RPI, who gets the credit for devising these questions.