1. Announcements and Overview
2. Review of material
3. Examples
4. Questions about homework and previous exams

1. **Announcements and Overview:** The first homework is due today in discussion! The second homework will be up soon if not already.

2. **Review of Material:** Recall the set $\Omega$ is the sample space of a given experiment. When we are discussing probability, we will always be referring to a single experiment. Any possible outcome for the experiment is listed in $\Omega$. In general, we are interested in either the probability of each outcome, or the probability of an event, which is a subset of $\Omega$. The modeling part of the problem comes in here, in developing the Probability Law, which assigns a probability to each event. Different assumptions about the system at hand can lead to different laws. However, once the probability law is established, calculating the probability of a given event is well defined, and is generally what we will be doing in this class. A probability law on $\Omega$ must satisfy certain properties, such as

$$P(\Omega) = 1$$

$$P(\emptyset) = 0$$

$$P(A \subset \Omega) \geq 0$$

$$P(A \cup B) = P(A) + P(B) \text{ if } A \text{ and } B \text{ are disjoint}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B).$$

A uniform probability model is perhaps one of the most common probability models out there. If we have a discrete finite uniform probability model which has $n$ outcomes, $s_n$, they all have the same probability of occurring, $P(s_i) = \frac{1}{n}$. Given some event $A \subset \Omega$, we can calculate quite easily $P(A) = \frac{|A|}{|\Omega|}$. An example of this could be flipping a coin $n$ times, and $A$ could be the event that $m \leq n$ heads are flipped. This differs significantly from a continuous probability model. An example of a continuous probability model could be a model of the action potential of a neuron in the brain, and $A$ could be whether or not the neuron spikes a number of times between a certain time frame. This probably shouldn’t be modeled by a uniform probability distribution, but rather something more sophisticated such as a Poisson distribution.

One last thing in section 1.2 that is notable is Bertrand’s paradox, which I will talk about in discussion briefly.

3. **Examples:** Show the "Inclusion Exclusion Principle",

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

To do this, group $A \cup B$ and use the property of the probability measure,

$$P((A \cup B) \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)$$

$$= P(A \cup B) + P(C) - P((A \cap C) \cup (B \cap C))$$

$$= P(A \cup B) + P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Next problem, prove $P(A \cap B) \geq P(A) + P(B) - 1$. To do this, just recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, and note that $-1 \leq -P(A \cup B)$. Then rearranging gives us

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1.$$
Next problem, prove that the real numbers between \([0, 1]\) are uncountable. Here, I’ll sketch Cantor’s diagonalization argument. Suppose that we can list all the real numbers between 0 and 1, meaning that we can assign a natural number to each real number in our interval (this is what we mean when we say a set is countable). Ignoring the non-uniqueness problem of decimal representations, our list will look like,

\[
1 : 0.a_1a_2a_3 \ldots \\
2 : 0.b_1b_2b_3 \ldots \\
3 : 0.c_1c_2c_3 \ldots \\
\vdots
\]

If we can find a real number that is not listed in this infinite list, then it must follow that we can not assign a positive integer to each real number in \([0, 1]\). We can construct such a number by choosing it to disagree with the \(n\)-th number at the \(n-\text{th}\) digit. Then it disagrees with each number in the list at least once (thus it isn’t listed already), and it is obviously a real number in \([0, 1]\). Therefore, we real numbers can’t be listed, and are not countable.

4. Questions on homework?

2nd Challenge Question of the month\(^1\): Consider a pyramid scheme initiated by a group of 5 people seeking to exploit a gullible population. Each person in the original group of 5 people sends out recruitment letters with their 5 names on a ”manager list” to 5 people within the population, enticing each of them to contribute money to each of the 5 names on the ”manager list” so that the letter recipient can participate in the scheme by deleting one name from the top of the ”manager list,” adding his name to the bottom of the ”manager list” and sending recruitment letters with this updated ”manager list” (again with 5 names on it) to 5 people. Suppose the population is so gullible that each person receiving one of these letters agrees to participate in the scheme and so eager that they immediately send out their 5 letters to new people. Then we can think of the pyramid scheme in terms of generations which act more or less in synchrony, with the first generation being the 5 initiators, the second generation being the people receiving letters from the first generation and sending out letters more or less at the same time to more people, which become the third generation, and so on. However, suppose that if any person already in the pyramid scheme receives a recruitment letter from someone in the same or later generation, the scales fall from their eyes and they realize the problem with pyramid schemes, and immediately publicize their realization, causing the pyramid scheme to collapse. Similarly, a person not already in the pyramid who receives more than one recruitment letter will also become suspicious, expose the pyramid scheme, and cause it to collapse. Construct a classical probability model based on the assumption that each new recruit to the pyramid will send their recruitment letters to 5 people selected completely at random from the population, except for obviously dumb choices like sending a recruitment letter to themselves or someone on the ”manager list” they received, or sending multiple recruitment letters to the same person.

Suppose the total size of the population is \(n\), and that the pyramid scheme has survived \(r\) generations. Based on your classical probability model, calculate the probability that the pyramid scheme will collapse at the next generation.

\(^1\)The challenge problems this quarter come from my first probability class with Peter Kramer at RPI, who gets the credit for devising these questions.