1. Announcements and Overview

Homework 5 is due on Friday. Go over homework 4 in detail. Mechanical Vibrations, Newton’s law, and more.

2. Review of material and examples:

(a) 18.3) Given \( \frac{d^2 x}{dt^2} = x - x^2 \), determine the possible equilibrium solutions and their stability.

Solution: This function \( f(x) = x - x^2 \) has zeros at \( x = 0 \) and \( x = 1 \). Therefore, we see that \( f'(0) = 1 \) and \( f'(1) = -1 \), so \( x = 1 \) is stable and \( x = 0 \) is unstable.

(b) 14.2) Consider a mass \( m \) at \( x(t)\hat{i} + y(t)\hat{j} \). \( L \) is the particle’s distance from the origin. Using polar coordinates, what is its velocity vector? Show that acceleration is,

\[
\frac{d^2 x}{dt^2} = \left( L \frac{d^2 \theta}{dt^2} + 2 \frac{dL}{dt} \frac{d\theta}{dt} \right) \frac{d\theta}{dt} + \left( \frac{d^2 L}{dt^2} - L \left( \frac{d\theta}{dt} \right)^2 \right) \hat{L}.
\]

Solution: Recall polar coordinates,

\[
x = L \cos(\theta), \quad y = L \sin(\theta)
\]

\[
L = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x)
\]

\[
\hat{L} = \cos(\theta)\hat{i} + \sin(\theta)\hat{j}
\]

\[
\frac{d\theta}{dt} = -\sin(\theta)\hat{i} + \cos(\theta)\hat{j}
\]

\[
\hat{i} = \cos(\theta)\hat{L} - \sin(\theta)\hat{\theta}
\]

\[
\hat{j} = \sin(\theta)\hat{L} + \cos(\theta)\hat{\theta}
\]

A word on the unit vector transforms. To find \( \hat{L} \) for example, one computes the vector \( \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} := \frac{d}{dt} x \). Likewise, \( \hat{\theta} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = -\sin(\theta)\hat{i} + \cos(\theta)\hat{j} \).

Let us write the velocity as \( v = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \). Because \( L = L(t) \) and \( \theta = \theta(t) \), we can use the chain rule,

\[
v = \frac{dL}{dt} \cos(\theta)\hat{i} - L \sin(\theta) \frac{d\theta}{dt} \hat{i} + \frac{dL}{dt} \sin(\theta)\hat{j} + L \cos(\theta) \frac{d\theta}{dt} \hat{j}.
\]

Using the identities we recalled, this is,

\[
v = \frac{dL}{dt} \hat{L} + L \frac{d\theta}{dt} \hat{\theta}.
\]

Differentiating \( x(t) \) twice, we see that

\[
\frac{d^2 x}{dt^2} = \left( \frac{d^2 L}{dt^2} \cos(\theta) - \frac{dL}{dt} \frac{d\theta}{dt} \sin(\theta) - \frac{dL}{dt} \frac{d\theta}{dt} \sin(\theta) - L \cos(\theta) \left( \frac{d\theta}{dt} \right)^2 - L \sin(\theta) \frac{d^2 \theta}{dt^2} \right) \hat{i}
\]
The problem is to consider a particle of mass \( m \) suspended between two identical springs. Show that the system is nonlinear, and find the steady state solution for a driving force \( F_0 \cos(\omega t) \).

In this problem, we will neglect gravity and vertical motion. When the particle is displaced from equilibrium, each spring exerts force \(-k(s - l)\) on the particle. Thus the net horizontal force is \( F = -2k(s - l) \sin(\theta) \). Note that \( \sin(\theta) = \frac{x}{s} = \frac{x}{\sqrt{l^2 + x^2}} \), and \( s = \sqrt{x^2 + l^2} \).

Therefore we see that the force is

\[
F = -\frac{2kx}{\sqrt{l^2 + x^2}} \cdot (\sqrt{l^2 + x^2} - l) = -2kx \left(1 - \frac{1}{\sqrt{1 + (x/l)^2}}\right). \quad (15)
\]

If we consider \((x/l)\) to be a small quantity, then we may expand the radial using the binomial expansion to get,

\[
F = -kl \left(\frac{x}{l}\right)^3 \left[1 - 3 \left(\frac{x}{l}\right)^2 + \cdots\right]. \quad (16)
\]

Ignoring all but the leading order term, we have \( F(x) = \frac{-k}{l^3} x^3 \), which is nonlinear. This is interesting because the system at hand is intrinsically nonlinear, even under a small horizontal motion, one can not escape the nonlinearity.

Now if we stretched each spring distance \( d \) to attach it to the mass at the equilibrium position, we find that the force is,

\[
F(x) = -2(kd/l)x - [k(l - d)/l^3]x^3. \quad (17)
\]

You won’t be expected to derive this, that derivation is beyond the scope of this course (though not terribly difficult, just not relevant). If we identify \( \epsilon' = -(l - d)/l^3 < 0 \) as a small parameter. Supposing that we have a driving force like \( F_0 \cos(\omega t) \), the equation of motion becomes,

\[
m x''(t) = -2(kd/l)x - [k(l - d)/l^3]x^3 + F_0 \cos(\omega t). \quad (18)
\]

Redefining some constants because I don’t feel like typing, let

\[
\epsilon = \frac{\epsilon'}{m}, \quad a = \frac{2kd}{ml}, \quad G = \frac{F_0}{m}, \quad (19)
\]

we have the equation

\[
x'' = -ax + \epsilon x^3 + G \cos(\omega t). \quad (20)
\]

Now we could nondimensionalize the problem to get rid of some constants (like \( \omega \)) but I will opt not to for brevity. This brings us back to perturbation methods. I will not work out the details here, but for completeness I will write down an approximate solution for \( x(t) \). Unfortunately up to this point, we have not gone over successive approximations, so this part should seem mysterious. Again, this is only being included for completeness.
Starting with the first order approximation that $x_1(t) = A \cos(\omega t)$, we can find a higher-order approximation $x_2(t)$ by plugging $x_1$ into the right side of the equation yielding,

$$x''_2(t) = -aA \cos(\omega t) + \epsilon A^3 \cos^3(\omega t) + G \cos(\omega t)$$

$$= -\left( aA - \frac{3}{4} \epsilon A^3 - G \right) \cos(\omega t) + \frac{1}{4} \epsilon A^3 \cos(3\omega t).$$  \hspace{1cm} (21)

Integrating this twice with zero initial conditions, you’ll find that

$$x_2 = \frac{1}{\omega^2} \left( aA - \frac{3}{4} \epsilon A^3 - G \right) \cos(\omega t) - \frac{\epsilon A^3}{36\omega^2} \cos(3\omega t).$$  \hspace{1cm} (22)

What is probably confusing is the notation. Note that in the asymptotic expansion portion of the class, we wrote that $x(t) \sim x_1(t) + \epsilon x_2(t) + \cdots$. The $x_1$ and $x_2$ used in that expansion are different than the $x_1$ and $x_2$ used above. In the method used above, its okay to think of $x_2$ as the same as the entire "2 term" expansion, whereas $x_1$ was just a leading order approximation. This part isn’t important in terms of this class, and I would be willing to talk about it further in office hours.

Figure 1: The mass spring system we are dealing with in item 3.

3. Questions on homework 5?