1. Announcements and Overview
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1. **Announcements and Overview**: Homework 3 is due on Friday. Up this week, nondimensionalization, simple perturbation problems, and questions on homework 3. I will not go over the singular perturbation problem but I included it for the interested student. Comments on Homework 2: People on average did much better (9.48/12). Mostly just algebra errors, presentation errors like not stapling their work still, and not showing work for the last part of the problem 3.

2. **Review of material and examples**:

(a) **Nondimensionalization**: Consider the following dimensional problem with $y$ being a one dimensional mass density,

$$y''(x) + 2\alpha y'(x) + 2\beta y(x) = 0, \ y(0) = 0, \ y\left(\frac{\alpha}{\beta}\right) = y_0, \ \alpha >> \beta > 0, \ x \in \left[0, \frac{\alpha}{\beta}\right]. \ (1)$$

Noting the dimensions,

$$[\alpha] = \frac{1}{L}, \ [\beta] = \frac{1}{L^2}, \ [x] = L, \ (2)$$

we can nondimensionalize this problem by letting $x = x_c X$, $y = y_c Y$, where $[x_c] = L$, $[X] = 1$. Plugging this in,

$$\frac{y_c}{x_c^2} Y''(X) + 2\alpha \frac{y_c}{x_c} Y'(X) + 2\beta y_c Y(X) = 0. \ (3)$$

We can form 2 dimensionless groups,

$$\frac{1}{\beta x_c^2} Y''(X) + 2\frac{\alpha}{\beta x_c} Y'(X) + 2 Y(X) = 0. \ (4)$$

At this point, it seems as if we have a lot of freedom on how to simplify this, and we do. One good choice may be letting $x_c = \frac{\alpha}{\beta}$, making the dimensionless group on the second term equal to 1. The first dimensionless group becomes $\frac{\beta}{\alpha}$ which is small by our original assumption that $\alpha >> \beta > 0$, so we shall rename this number $\epsilon$, standing in for our small dimensionless group. Letting $y_c = y_0$, the problem becomes a dimensionless one with a small parameter,

$$\epsilon Y''(X) + 2Y'(X) + 2Y(X) = 0, \ \ 0 < X < 1, \ Y(0) = 0, \ Y(1) = 1, \ x = \frac{\alpha}{\beta} X, \ y = y_0 Y. \ (5)$$

(b) **Approximate roots of algebraic equation**: The equation handed to us is the following,

$$x^2 + (1 - 4\epsilon)x - \sqrt{1 + 4\epsilon} = 0. \ (6)$$

This is a quadratic equation which we all know how to solve analytically, but for the purposes of getting used to "matching" terms of the same order, we will find an asymptotic approximation of the solution. To begin, assume that the roots are of the form $x \sim x_0 + \epsilon^a x_1 + \cdots$, where hopefully the symbol $\sim$ has been discussed in lecture previously. This is already an assumption, we have assumed that the leading order term of the roots is $\mathcal{O}(1)$, or that it does not depend on $\epsilon$ in a significant way. This happens to be correct, but this is not always true. In those cases, starting with $x \sim \epsilon^a x_0 + \epsilon^b x_1 + \cdots$, $a < b < \cdots$ would be more appropriate. Nevertheless, we will drop
all but the largest two terms in our expansion, the first two terms. Plugging in \( x = (x_0 + \epsilon^a x_1) \)
into the original equation, one has

\[
0 = (x_0 + \epsilon^a x_1)^2 + (1 - 4\epsilon)(x_0 + \epsilon^a x_1) - \sqrt{1 + 4\epsilon} \tag{7}
\]

\[
= x_0^2 + 2\epsilon^a x_0 x_1 + \epsilon^{2a} x_1^2 + x_0 + \epsilon^a x_1 - 4x_0\epsilon - 4\epsilon^{a+1} x_1 - 1 - 2\epsilon \tag{8}
\]

Note that the binomial expansion was used to simplify the last term of the equation, namely

\[
\sqrt{1 + 4\epsilon} = 1 + \frac{\epsilon}{2} + O(\epsilon^2).
\]

The resulting equation appears to be complicated, but it is not. At this stage, one wants to take all the terms of the highest order \( O(\epsilon^0) \), and let them cancel each other out. In other words, we have

\[
\mathcal{O}(1): \quad x_0^2 + x_0 - 1 = 0 \tag{9}
\]

\[
x_0 = \frac{-1 \pm \sqrt{5}}{2}. \tag{10}
\]

Now to match the next order terms. We see that we have several orders of \( \epsilon \) in our equation at this point, we have terms with an \( \epsilon \) out front, \( \epsilon^2 \), \( \epsilon^{2a} \), etc. We are free to choose \( a \) so that we can cancel out the terms with an \( \epsilon \) attached, under the assumption that our choice of \( a \) leaves all the other terms at a higher order (meaning we will be able to ignore them). What do I mean? Let us try to pick \( a = 1 \). Then we have terms of order \( \epsilon, \epsilon^2, \ldots \). Just as we did for the \( \mathcal{O}(1) \) problem, we can set equal the \( O(\epsilon) \) terms, and ignore the higher order terms. Collecting the terms with an \( \epsilon \) out front gives us,

\[
\mathcal{O}(\epsilon): \quad 2x_0 x_1 + x_1 - 4x_0 - 2 = 0 \tag{11}
\]

\[
x_1 = \frac{4x_0 + 2}{2x_0 + 1}; \tag{12}
\]

\[
x_1 = 2. \tag{13}
\]

In the last step, I plugged in \( x_0 \) equal to what was found in the \( \mathcal{O}(1) \) equation. Therefore, we have found a two-term approximation for each root, namely

\[
x = \frac{-1 \pm \sqrt{5}}{2} + 2\epsilon. \tag{14}
\]

**Note:** We see that as \( \epsilon \) shrinks, our approximation gets better and better (and it is exactly correct for \( \epsilon = 0 \)). This is the nature of all asymptotic approximations. Finding more terms in your expansion does not always give you a better approximation. If you wish to improve your approximation, then one must shrink \( \epsilon \). This is not something that you can just do in real physical problems, but that is the nature of making approximations. note

(c) **Regular Perturbation Problem:** After nondimensionalizing an ODE that arises from a physical problem, one many times is left with a small parameter in the differential equation. If this perturbation term is not the highest order derivative in the the equation, than we call this a regular perturbation problem. The next example will demonstrate some difficulties that can arise when \( \epsilon \) is attached to the highest derivative. However, we will consider the nondimensionalized problem,

\[
\frac{dv}{dt} + \epsilon v^2 + v = 0, \quad 0 < t, \quad v(0) = 1. \tag{15}
\]

If we would like to find a two term approximation, we begin by letting \( v(t) \sim v_0(t) + \epsilon^a v_1(t) \). Plugging this into the DE, one arrives at

\[
v'_0 + \epsilon^a v'_1 + \epsilon^2 v_0^2 + 2v_0 v_1 \epsilon^a + O(\epsilon^{2a}) + v_0 + \epsilon^a v_1 = 0. \tag{16}
\]

Notice that I purposely left out the term that was higher order in \( \epsilon \), basically just to save me the work of writing it down. Going about things the analogous way to the algebraic problem, we will look at the largest terms, or the terms with the lowest degree of \( \epsilon \) appearing \( (\epsilon^0 \) to be exact:

\[
\mathcal{O}(1): \quad v'_0(t) + v_0(t) = 0, \quad v_0(0) = 1 \tag{17}
\]

\[
v_0(t) = e^{-t}. \tag{18}
\]
Now, we must match the next order of $\epsilon$. Similar to the first problem, we choose $a = 1$ to do this. Then the $O(\epsilon)$ problem becomes,

\[ O(\epsilon) : \quad v_1''(t) + v_1^2 + v_1 = 0, \quad v_1(0) = 0 \quad \text{because there is no $\epsilon$ dependence in the I.C.} \]  
\[ v_1' + v_1 = -e^{-2t} \]  
\[ v_1(t) = e^{-2t} + Ce^{-t} \]  
\[ = e^{-2t} - e^{-t}. \]  

The last step is applying the initial condition. Therefore, we have found a two term approximation for $v(t)$,

\[ v(t) \sim e^{-t} + \epsilon(e^{-2t} - e^{-t}), \]  
where the approximation becomes better and better as $\epsilon \to 0$.

(d) **Singular Perturbation Problem** *(NOTE: This is not something needed for the upcoming tests or homeworks!)*: The problem to be considered is

\[ \epsilon y'' + 2y' + 2y = 0, \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = 1. \]  

We call this a *singular perturbation problem* because there is a small parameter $\epsilon$ attached to the highest derivative $y''$. Why is this significant? Because as $\epsilon \to 0$, the ODE changes to a first order ODE. A first order ODE can not solve both boundary conditions in general. In fact, this should hint to oneself that there is a small region (perhaps of $O(\epsilon)$ size) where the solution rapidly changes in order to accomadate for both boundary conditions. This region of rapid transition is called a *boundary layer*, because it is at the boundary in this case. Note that in more complicated problems, interior layers and multiple boundary layers can arise.

But perhaps we are naive to this boundary layer and we try to go about things the normal way. Letting $y(x) \sim y_0 + \epsilon y_1(x)$, the equation becomes,

\[ \epsilon(y_0'' + \epsilon y_1'') + 2(y_0' + \epsilon y_1') + 2(y_0 + \epsilon y_1) = 0, \quad y_0(0) = 0, \quad y_0(1) = 1. \]  

Only dealing with the terms of $O(1)$, one has,

\[ O(1) : \quad 2y_0' + 2y_0 = 0 \quad \implies \quad y_0(x) = e^{1-x}, \]  

after applying the initial condition at $x = 1$. Note that we have two initial conditions, but a first order ODE needs only one condition to be well-posed. Although I am only planning on keeping $y_0(x)$ in our asymptotic expansion, I’ll compute $y_1(x)$ for completeness,

\[ O(\epsilon) : \quad y_0'' + 2y_0' + 2y_0 = 0, \quad y_1(1) = 0 \quad \implies \quad y_1(x) = \frac{(1 - x)e^{1-x}}{2}. \]  

Now, we have that our $y_0(x)$ is equal to $e^1$ at $x = 0$. This is a problem, considering that our boundary condition at $x = 0$ says that $y(x)$ should be zero. For this reason, we will look for a so-called "inner-layer" solution, denoted by $Y(X)$, where $X$ will be our inner layer coordinate. Due to this naming convention, $y_0(x)$ will be called the "outer solution", because it is outside the boundary layer. Because there is some sort of rapid variation going on in a small region around $x = 0$, we will stretch this coordinate as follows, define

\[ X = \frac{x}{\epsilon^\gamma}. \]

We see that when $x$ is very small, $X$ is an $O(1)$ quantity. Using the chain rule, the derivative transforms like,

\[ \frac{d}{dx} = \frac{dX}{dx} \frac{d}{dX} = \frac{1}{\epsilon^\gamma} \frac{d}{dX}. \]  

The scaled equation around $x = 0$ is then,

\[ \epsilon^{1-2\gamma}Y'' + 2e^{-\gamma}Y' + 2Y = 0. \]  

\[ \]
We need either two of these terms to be on the same order of magnitude, with the third term being of higher order, or we need all three of these terms to be "of the same size". We see that no value of $\gamma$ exists that will make all three of these terms the same order in $\epsilon$, so maybe we could guess that the first and third terms will be the same order. This would mean $\gamma = 1/2$, but then we see that the second term is actually much larger than the other two! This is because it is $O(\epsilon^{-1/2})$, as compared to the other terms being $O(1)$. Therefore, our only option left is that the first and the second terms are the same size. This means $\gamma = 1$, and the equation is then

$$Y'' + 2Y' + 2\epsilon Y = 0. \quad (30)$$

Again, note that the last term is higher order than the other two. Letting $Y(X) \sim Y_0(X) + \cdots$, we can find a leading order approximation for the inner solution,

$$O(1) : \quad Y_{0''} + 2Y_{0'} = 0, \quad Y_0(0) = 0 \implies Y_0(X) = A(1 - e^{-2X}). \quad (31)$$

Now that we have the approximate behavior of the solution in both the inner and outer regions, we can glue them together. How is this done? In this simple case, it is sufficient to just assert that $\lim_{x \to 0} y_0(x) = \lim_{X \to \infty} Y_0(X)$. (Why is the second limit going toward infinity?) Computing these limits, its not hard to see that $A = e$. Then to get a final composite solution, we can write

$$y(x) \sim y_0(x) + Y_0(X) - \lim_{x \to 0} y_0(x). \quad (32)$$

The common value at the end must be subtracted out because both $y_0$ and $Y_0$ will contribute that term. Upon doing so, one finds that

$$y(x) \sim e^{1-x} - e^{1-2x}. \quad (33)$$

It is instructive to look at graphs of the exact solution (which the reader should be able to explicitly compute), and the approximate solution for small $\epsilon$. One finds that the approximate solution is remarkably accurate.

3. Questions on homework 3?