1. Introductions

2. Discussion Structure

3. Expectations

4. Questions

5. Review

1. Introductions and Background

2. Discussion Structure & Office Hours

3. Expectations

(a) Grading: I am responsible for partially grading exams and for grading all homeworks. Homework grading is based on correctness and clarity. If I can not read your solution, I can not determine its correctness accurately and will mark it low. This low mark may be redeemed in office hours if it is possible to clarify what was written without changing the original answer. That being said, I tend to grade the neater homeworks when I’m in a better mood, drinking coffee, hanging out with my dog. Accordingly, I tend to grade the sloppier homeworks when I’m hungry.

(b) Discussion: Attendance is never mandatory. There will be no exams given during discussion, so missing discussion will not have any direct impact upon your final grade. However, it is a general trend that those who attend discussion receive higher final grades compared to those who do not. This is likely due to reinforcing the material, as well as helpful discussion about homework. If you choose to attend, please do not get up and leave (unless its an emergency) between the 5-49 minute marks, as it is disruptive to your fellow peers. Questions may be called out.

(c) Office Hours: Office hours operate on a first come, first serve basis. However, sometimes this rule is broken when many students are present in order to maximize the benefit of office hours for all students. If your question goes unanswered at office hours, I can almost always accommodate and set up another time to meet if necessary, or just email if its a quick question.

(d) Online Notes: I usually try to post notes before or soon after each discussion. Don’t take this for granted, it is not part of my duty as a teaching assistant. However, I’ve found that many students like this and it benefits the students as a whole, so I try my best to keep up with posting notes. If attendance for discussion becomes sufficiently low, I will stop posting notes. The purpose of these notes are not so one can skip every single discussion, but so those who do not wish to take notes in discussion do not have to, and still have the material to look over later.
4. Questions

5. Review We assume proficiency in basic algebra, calculus, and multivariable calculus. We also assume some familiarity with basic differential equations, methods of how to solve separable odes, linear constant coefficient odes, and inhomogeneous 1st and 2nd order odes. These skills will be essential in this class, so most of it will be reviewed in detail. Previous exposure to linear algebra, eigenvalue problems, and physics won’t hurt either, but none are necessary for success in this class.

Below, I have included a review of essentials that will reoccur over and over again.

(a) Taylor’s Theorem:
Statement of the theorem:
Let \( f \in C^n(\mathbb{R}) \), the space of functions that are \( n \) times differentiable. Let \( f : \mathbb{R} \to \mathbb{R} \), then there exists a point \( \xi \) in the domain such that

\[
f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \cdots + \frac{(x - a)^n}{(n-1)!}f^{n-1}(a) + R_n,
\]

where the most useful form of the remainder is written as

\[
R_n = \frac{(x - a)^n}{n!}f^{(n)}(\xi).
\]

In this class, many of the proofs will be omitted. This is not because they are unimportant or uninteresting, but because the purpose of this class is to introduce students to how these theorems can be applied in analyzing mathematical models of physical phenomena. In higher dimensions, Taylor’s theorem will be central for linearizing nonlinear functions.

(b) Vector Notation: For the remainder of this class, I will be using a little bit of notation which may be unfamiliar to some. The reason for this is not to hide calculations or confuse people, but rather to simplify the (possible repetitive) work being done when doing a mathematical operation, such as taking a multidimensional derivative. Good notation sets the brain free of all unnecessary work, and allows it to concentrate on more advanced problems, and in effect increases the mental power of the race (who said that?)\(^1\)

A vector quantity will be denoted by \textbf{boldface}, and this could be either a function \( f : \mathbb{R}^n \to \mathbb{R}^m \) or a coordinate \( x \in \mathbb{R}^n \). In other words, \( x = (x_1, x_2, \ldots, x_n) \) and

\[
f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{pmatrix}.
\]

One important piece of notation that has undoubtedly been seen before if the gradient operator for \( f : \mathbb{R}^n \to \mathbb{R} \),

\[
\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.
\]

Another important, but less familiar, derivative operator \( D \) that is for a vector valued function. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \), then

\[
Df(x) = \begin{pmatrix} \nabla f_1(x) \\ \nabla f_2(x) \\ \vdots \\ \nabla f_m(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.
\]

(c) Linear constant coefficient homogeneous \( n \)-th order ODEs: These equations are of the form

\[
L[y] = y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1y'(x) + a_0y(x) = 0,
\]

\(^1\)Alfred Whitehead, an English mathematician of the 20th century.
where we say that \( \mathcal{L} \) is a linear differential operator acting on \( y(x) \). The coefficients, \( a_i, i = 0, 1, \ldots, n - 1 \), are constants here.

Here’s a common example that one can not escape,
\[
y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{7}
\]
For linear constant coefficient ODEs, make the ansatz that \( y(x) = e^{rx} \), and plug it in
\[
e^{rx}(r^2 + \omega^2) = 0 \tag{8}
e^{rx} \neq 0 \implies r^2 = -\omega^2 \tag{9}
r = \pm i\omega. \tag{10}
\]
A second order ODE has two solutions, \( y(x) = e^{i\omega x} \) and \( e^{-i\omega x} \), and its not difficult to see that writing down \( y \) as a linear combination of the two is also a solution,
\[
y(x) = Ay_1(x) + By_2(x) = Ae^{i\omega x} + Be^{-i\omega x}. \tag{11}
\]
Solve for the unknown constants using the initial conditions,
\[
y(0) = 1 = Ae^0 + Be^0 \implies A + B = 1 \tag{12}
y'(0) = 3 = Ai\omega e^{i\omega x} - i\omega Be^{-i\omega x} \bigg|_{x=0} \tag{13}
3 = Ai\omega e^0 - i\omega(1 - A)e^0 \tag{14}
A = \frac{3 + i\omega}{2i\omega} \tag{15}
B = 1 - A. \tag{16}
\]
This is a slightly dirtier way compared to writing \( Ae^{i\omega x} + Be^{-i\omega x} = C \cos(\omega x) + D \sin(\omega x) \), but nevertheless, they give the same thing in the end.

**Linear inhomogeneous first order ODEs:** Consider an ODE of the form,
\[
\frac{dy}{dx} + P(x)y(x) = Q(x). \tag{17}
\]
The one way that always works is using an integrating factor. Depending on the form of \( P \) and \( Q \), other methods (separable, the exponential ansatz above, etc.) may work. However, an integrating factor will always work if used correctly. Focusing only on the left hand side, we see that this side almost looks like a product rule. Recall the product rule,
\[
(R(x)y(x))' = R(x)y'(x) + R'(x)y(x). \tag{18}
\]
If we multiply the entire ODE by the right value, the left hand side will look exactly like this, and that value is \( e^\int P(x) \, dx \). Let’s look at why,
\[
e^\int P(x) \, dx \frac{dy}{dx} + P(x)e^\int P(x) \, dx y(x) = Q(x)e^\int P(x) \, dx, \tag{19}
\]
considering the left hand side still, we see that
\[
\left[e^\int P(x) \, dx y(x)\right]' = e^\int P(x) \, dx \frac{dy}{dx} + P(x)e^\int P(x) \, dx y(x). \tag{20}
\]
Look at it again, do this product rule out by hand, convince yourself of this and you’ll have an easier time in life. Then we have a problem we can solve straight up,
\[
\frac{d}{dx} \left[e^\int P(x) \, dx y(x)\right] = Q(x)e^\int P(x) \, dx. \tag{21}
\]
\footnote{If this type of guessing makes you uncomfortable, I could give a more convincing argument in office hours for why this should be the form of the solution.}
Integrate,
\[ e^{\int P(x) \, dx} y(x) = C_1 + \int^x Q(\xi) e^{\int^x P(X) \, dX} \, d\xi \]  
and isolate \( y(x) \),
\[ y(x) = C_1 e^{\int P(x) \, dx} + \int^x Q(\xi) e^{\int^x P(X) \, dX - \int^P P(x) \, dx} \, d\xi, \tag{23} \]
where we see that the effect of the inhomogeneous term comes in as a convolution with some integral kernel.\(^3\) There are many cool results in higher mathematics that take this form.

A concrete example,
\[ y' - \frac{1}{x} y = x^2, \quad y(1) = 2. \tag{24} \]
Identify \( P(x) = -\frac{1}{x} \), and \( e^{\int^{1/x} \, dx} = e^{-\ln(x)} = x^{-1} \), so we see that
\[ \frac{y'(x)}{x} - \frac{y(x)}{x^2} = x \tag{25} \]
is of just the right form to write it as a product rule,
\[ \frac{d}{dx} \left[ \frac{y(x)}{x} \right] = x, \tag{26} \]
and then go through the steps,
\[ \frac{y(x)}{x} = C_1 + \int x \, dx \tag{27} \]
\[ y(x) = C_1 x + x \left( \frac{1}{2} x^2 \right) \tag{28} \]
\[ y(1) = 2 = 1C_1 + \frac{1}{2}(1)^3 \tag{29} \]
\[ C_1 = \frac{3}{2}, \tag{30} \]
\[ y(x) = \frac{3x + x^3}{2}. \tag{31} \]

**Linear inhomogeneous second order ODEs:** A prototypical ODE of this family would be of the form,
\[ y''(t) + p(t)y'(t) + q(t)y(t) = F(t). \tag{32} \]
Recall that the solution is of the form \( y(t) = y_H(t) + y_I(t) \), where \( y_H \) satisfies \( y''_H(t) + p(t)y'_H(t) + q(t)y_H(t) = 0 \), and \( y_I \) can be any function that satisfies the differential equation. By analogy, in linear algebra, finding the "total solution" to \( Ax = b \) consists of finding a particular solution \( x_p \) plus any linear combination of nullspace solutions. This is because if \( x = x_p + x_1 + \cdots + x_n \), where \( x_i \) satisfy \( Ax_i = 0 \), we have that \( Ax = A(x_p + x_1 + \cdots + x_n) = Ax_p + Ax_1 + \cdots + Ax_n = b + 0 + \cdots + 0 = b \). The case is analogous when we have a linear differential operator \( L[y] := y''(t) + p(t)y'(t) + q(t)y(t) \) equal to some right hand side \( F \). The solution will be a particular solution, plus an arbitrary linear combination of nullspace (homogeneous) solutions.

Before I begin to solve an equation of this form with constant coefficients, I will mention that there is a more involved but sure-fire way to solve these types of equations when the coefficients are constant (and this can be generalized to n-th order equations easily), but it is inefficient if \( F(t) \) is a "nice" function. This method is called variation of parameters, and I am willing to go over it in detail in office hours. The method I will use is called *The method of undetermined coefficients*, or in other words, guessing the form of the particular solution, and then checking to see if it works.

\(^3\)Duhamel’s principle in PDE’s is very reminiscent of this.
Consider an instance of the constant coefficient case

\[ y'' + y = \sin(kt), \quad k \neq 1. \]  

(33)

The homogeneous solution can be found by the previous techniques to be \( y_H(t) = c_1 \sin(t) + c_2 \cos(t) \), where you can think of \( \sin(t) \) and \( \cos(t) \) as the basis vectors for the nullspace of the linear operator \( L[y] = \left( \frac{d^2}{dt^2} + 1 \right)y(t) \). Using the method of undetermined coefficients, we have a reasonable guess for the form of the particular solution,

\[ y_I(t) = A \sin(kt) + B \cos(kt). \]  

(34)

Plugging this back into the ODE, we have

\[ (1 - k^2)(A \sin(kt) + B \cos(kt)) = \sin(kt). \]  

(35)

Great, if we set \( B = 0 \) and \( A = \frac{1}{1 - k^2} \), we have that \( y_I = \frac{\sin(kt)}{1 - k^2} \) solves the ODE when \( k \neq 1 \).

Then our total solution is

\[ y(t) = c_1 \sin(t) + c_2 \cos(t) + \frac{\sin(kt)}{1 - k^2}. \]  

(36)

Things get complicated when \( k = 1 \) because then the forcing function is the same as the solutions to the homogeneous equation. Guessing any function of that form would then be annihilated by the ODE, and therefore couldn’t equal the right hand side. Without going into detail, one typically appends a \( t \) onto what the guess would have been, e.g. if \( k = 1 \), I would guess the form,

\[ y_I(t) = At \sin(t) + Bt \cos(t). \]  

(37)

The reader is encouraged to see if this works.

(d) **System of ODEs:** We will consider the ODE system,

\[
\begin{align*}
x_1'(t) &= 3x_1(t) + x_2(t) \\
x_2'(t) &= x_1(t) + 3x_2(t),
\end{align*}
\]

(38) (39)

One can write this in its more suggestive vector form,

\[ \mathbf{x}'(t) = A\mathbf{x}, \]  

(40)

where

\[ A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}. \]  

(41)

In order to uncouple the system, and hence make the system solvable, we need to find the eigenvalues. The eigenvalues are \( \lambda_1 = 4 \) and \( \lambda_2 = 2 \). If we let \( \mathbf{e}_1 \) be the eigenvector corresponding to \( \lambda_1 \) and \( \mathbf{e}_2 \) the eigenvector corresponding to eigenvalue \( \lambda_2 \), we can write the solution to the system as,

\[ \mathbf{x} = A\mathbf{e}_1 e^{\lambda_1 t} + B\mathbf{e}_2 e^{\lambda_2 t}. \]  

(42)

Here, vectors \( A \) and \( B \) are determined from initial conditions if provided. Thus, once one determines the eigenvalues by solving \( (A - \lambda I)\mathbf{e}_i = \mathbf{0} \), one has

\[
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + B \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.
\]

(43)

We can look at the **phase plane** of the system, or \( x_1 \) plotted against \( x_2 \). Using the initial condition as the starting point of the system’s trajectory through the phase plane, we can visualize how the system will evolve with respect to time.

Note that this is not the only way to solve such a system. In particular, the matrix exponential is one notable technique for solving systems where \( A \) is diagonalizable, and for lower order systems, one can many times us substitution to get a higher order ODE (2nd order in this case), and then solve that instead.

As a final note, I will attempt to compile a list of useful resources (web pages, books, course notes, etc.) that may be helpful for this class, and keep it updated on my website.