TOPICS
1. Linearizing a System and solving
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DISCUSSION

1. **Linearizing a System and Solving:** The system we wish to solve is,

\[
x'(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} -x_1^2 + x_2 + 1 \\ 2x_1^2 - x_1x_2 + 2x_2 - 2 \end{pmatrix}.
\]

We immediately see that these are two nonlinear coupled equations, and therefore unless we have a very special problem on our hands (which we don’t), we won’t be able to find exact analytical solutions. However, what we can do is find the equilibrium point(s), linearize about these points, and determine the behavior of the system near those equilibrium points. To find the equilibrium point, or in other words, the point where the system is not changing with respect to time, this amounts to setting the derivatives equal to zero and solving for \( x_1, x_2 \).

One can see that \( x_E = (x_1, x_2) = (1, 0) \) is an equilibrium point.

Now to linearize about this point, we write,

\[
f(x) = f(x_E) + \left. \frac{\partial f}{\partial x} \right|_{x_E} (x - x_E).
\]

Here, \( f, x, \) and \( x_E \) are vector quantities, and the partial derivative is another way of writing the Jacobian matrix,

\[
\frac{\partial f}{\partial x} := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}.
\]

Therefore, since \( f(x_E) = 0 \), we have that around the equilibrium point,

\[
x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \left. \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \right|_{x_E} \begin{pmatrix} x_1(t) - 1 \\ x_2(t) \end{pmatrix}.
\]

We see that the Jacobian is,

\[
\frac{\partial f}{\partial x} \bigg|_{x_E} = \begin{pmatrix} -2x_1 & 1 \\ 4x_1 - x_2 & -x_1 + 2 \end{pmatrix}_{(1,0)} = \begin{pmatrix} -2 & 1 \\ 4 & 1 \end{pmatrix}.
\]

Letting the vector \( y(t) = x(t) - x_E \), we have that \( y'(t) = x'(t) \) and

\[
y'(t) = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} y(t),
\]

at least near the equilibrium point (which is now centered at the origin in our new coordinate \( y \)). To solve such a system, we could use the matrix exponential, we could use substitution to uncouple the
equations and obtain a 2nd order linear ODE, or we could use linear algebra and say that the solution is,

\[ y(t) = c_1 l_1 e^{\lambda_1 t} + c_2 l_2 e^{\lambda_2 t}, \]  

(7)

where \( c_1 \) and \( c_2 \) are scalar constants determined by the initial conditions, \( l_1, l_2 \) are eigenvectors of the Jacobian, and \( \lambda_1, \lambda_2 \) are their corresponding eigenvalues. If you plug this into the ODE for \( y'(t) \), one sees that this solution satisfies the ODE and is therefore a solution. Without going through the steps, we can readily check that \( \lambda_1 = 2, \lambda_2 = -3 \), and

\[ l_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad l_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \]  

(8)

This gives us that around the equilibirum point, we have that the system behaves approximately like

\[ y(t) = c_1 \left( \frac{1}{4} \right) e^{2t} + c_2 \left( \frac{1}{-1} \right) e^{-3t}. \]  

(9)

What this says is that in the direction of \( l_1 \) in the \( x_1, x_2 \) plane, the system tends to grow with \( t \), and in the \( l_2 \) direction, the system will tend to decay. Furthermore, if the system were to start exactly on one of these eigenvectors (say \( l_2 \) for example), then \( c_1 = 0 \) and the solution will simply decay toward the equilibirum point as \( t \) increases. See the later sections to classify this behavior (its called a saddle point).

2. **Phase Portrait:** When the Dutch physicist Balthasar van der Pol was studying electronic circuits\(^1\), a rather interesting differential equation arose which subsequently was named after him, *the Van der Pol equation*,

\[ x''(t) - \mu (1 - x^2) x'(t) + x = 0. \]  

(10)

Clearly this is a difficult ODE to solve, but as we discussed last time, any time we have a higher order ODE, we can instead write down a system of coupled first order ODEs that represent the same equation. Note that the dimension of the ODE system will be the degree of the original ODE.

Here however, we will consider an ODE system that exhibits very similar behavior as the Van Der Pol equation. Consider the system,

\[ \frac{du}{dt} = -v \]  

(11)

\[ \epsilon \frac{dv}{dt} = u + \frac{\lambda}{3} (v - v^3). \]  

(12)

Here, it is assumed that \( \epsilon > 0 \) is small, and \( \lambda \neq 0 \). The solution to this system gives us the solution to an ODE that looks very much like the Van der Pol equation, and therefore is of interest to us. However, solving a system of nonlinear first order ODEs is beyond the scope of this class, so we will have to be satisfied with determining the qualitative behavior of this system. We will do this by plotting the phase portrait, but first a definition.

**Definition:** Given an equation \( \frac{dx(t)}{dt} = f(x, y, t) \), we define the **x-nullcline** to be the curve in xyt-space such that \( \frac{dx}{dt} = 0 \).

**Examples:** For the system above, we have that the u-nullcline is just the line \( v = 0 \) and the v-nullcline is the curve \( u = -\frac{\lambda}{3} (v - v^3) \).

The steps to plotting a phase portrait are as follows:

(a) Find critical points (and possibly classify the critical points as discussed in the next section).

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\(^1\)This equation has also been used to model action potentials in neurons!
(b) Plot the nullclines and determine the slope across each nullcline.

(c) Divide the phase plane into distinct regions and find the slope in those regions.

This will all become clear hopefully by the time this example is finished. Starting with step 1, we see that there is only one critical point to this system, namely \((u,v) = (0,0)\). If we linearize the system about this point, our Jacobian matrix is,

\[
\frac{\partial f}{\partial x}
\bigg|_{(0,0)} = J(0,0) = \begin{pmatrix} \nabla f_1(0,0) \\ \nabla f_2(0,0) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{1}{\epsilon} & \frac{\lambda}{\epsilon^3} \end{pmatrix}
\] (13)

The eigenvalues, \(e\), of this matrix are

\[
e_{1,2} = \frac{\text{tr} \pm \sqrt{\text{tr}^2 - 4\det}}{2} = \frac{1}{2\epsilon}(\lambda \pm \sqrt{\lambda^2 - 4\epsilon}).
\] (14)

If \(\lambda < 0\), then \(\text{Re}(e_{1,2}) < 0\) means the system is stable. As \(\lambda\) becomes positive however, \(e_{1,2}\) become complex, and one can also see that at \(\lambda = 0\), \(\frac{d}{d\lambda} \text{Re}(e_{1,2}) \neq 0\), which are the conditions necessary for a so-called Hopf Bifurcation. You will not be tested on what this is, but for the interested student, one can read about the interesting implications for what this means as the system goes toward the unstable regime.

Enough of that, now we are on step 2 - plotting the nullclines. This is just an exercise in plotting functions. Next we are interested in determining at least approximately what the vector field on the uv-plane looks like. This will then give us an idea of how the system’s trajectory will evolve over time. To do so, we divide our two equations,

\[
\frac{du}{dv} = \frac{du}{dt} = \frac{-ev}{u + \frac{1}{3}(v - v^3)}.
\] (15)

We see that on the u-nullcline, we have \(v = 0\) and therefore \(\frac{du}{dv} = 0\), so the slope field is horizontal across this nullcline. Likewise, when we are on the v-nullcline, we have that \(\frac{du}{dv} \rightarrow \infty\), meaning it is a vertical slope through this nullcline. What this means is that if we plot the trajectory of the system, it must pass through the u-nullcline going horizontal and pass through the v-nullcline vertically.

Last, step 3 is to identify distinct regions. These regions are denoted \(I, II, III, IV\) in the figure below, and these regions are not quadrants, but regions separated by the nullclines. To find the slope in region \(I\), note that we have \(-ev < 0\) because \(v > 0\), and we are above the v-nullcline, so \(u + \frac{1}{3}(v - v^3) > 0\). This implies that the slope \(\frac{du}{dv} < 0\). Likewise, in region \(II\), we have that \(-ev > 0\) and \(u + \frac{1}{3}(v - v^3) > 0\), meaning that \(\frac{du}{dv} > 0\). This can then be done in a similar fashion for the remaining regions. The arrows on the plot below show what this means pictorially. What we have determined is that if the initial condition of the system lives in region \(I\) at \(t = 0\), it will evolve over time to move right and downward, then going vertical as it hits the v-nullcline, and entering into region \(IV\). This is demonstrated by letting the initial condition of the system be at point \(a\) labeled on the plot. As one can imagine the system changing with time, the system heads first to point \(b\), then \(c \rightarrow d \rightarrow e\), and then back to \(b\)!

Why is this the trajectory? Because it must follow the slope field, and because \(v' = \frac{1}{3}(u + \frac{1}{3}(v - v^3))\), we have that the change in \(v\) with respect to time is very large (\(\epsilon\) is small here), so the system will move mostly in the \(v\)-axis direction. When it is on a nullcline, it will follow the nullcline as far as it can until it is forced off of it in order to stay consistent with the slope field. This type of behavior is called a limit cycle.
3. **Classifying Critical Points in 2D:** The idea behind classifying critical points for 2D systems hinges on the observations that in some appropriate basis, the linearized problem

\[
x'(t) = Ax(t)
\]

(16) is diagonal, and hence uncoupled. Because the eigenvalues of the Jacobian lie on the diagonal, we expect that the qualitative behavior of the solutions \( x_1(t) = Ce^{\lambda_1 t} \) and \( x_2(t) = De^{\lambda_2 t} \) should be a solely a function of the eigenvalues of the Jacobian matrix. In general, we will find that \( \lambda \in \mathbb{C} \), and that the real part of the eigenvalues will determine the growth or decay of \( x_1, x_2 \), and the imaginary part of the eigenvalues will determine any oscillatory properties of the solution, because \( e^{ix} = \cos(x) + i \sin(x) \).

With this in mind, we say that if the real part of any eigenvalue is positive, the system is unstable because the system will blow up in time. If the real parts of all the eigenvalues are nonpositive, the solution will be stable (or asymptotically stable if they are all strictly negative). If the imaginary parts of the eigenvalues are nonzero, the behavior of the system will be oscillatory as well as possibly growing or decaying with time.

As will be discussed in actual discussion, we can define certain equilibrium points to be sources (unstable nodes), sinks (stable nodes), stable cycles, decaying/growing spirals, degenerate nodes, and saddle points.

4. **Homework Questions:**